Answers

CHAPTER 1

EXERCISE 1.1

1 **a** $-18x^5$ **b** $-x^{-\frac{9}{4}} = -\frac{1}{x^{\frac{9}{4}}}$ **c** $3x^2 + \frac{\sqrt{5}}{x^{\frac{2}{3}}}$

2 **a** $f'(x) = 6x^2 - 2x, f'(-2) = 28$

b $f'(x) = 2x + 1$

$f'(1) =$ undefined (since $f(x)$ is not defined at $x = 1$)

c $f'(x) = 2 + 9x^2$ $f'\left(\frac{1}{8}\right) = 2\frac{9}{64}$

3 $15x^2$

4 $16\frac{1}{2}$

5 **a** $\frac{\frac{1}{2}}{x^{\frac{1}{2}}} + \frac{\frac{1}{6}}{x^{\frac{5}{6}}} + \frac{\frac{1}{4}}{x^{\frac{3}{4}}}$

b Proof: see worked solutions

6 $\frac{-3(2x+1)}{x^4}$

7 $f'(x) = x^{a-1}$

$4^{a-1} = 16 \Rightarrow a = 3$

8 $3 - 15x^2$

9 $a = -1$ and $b = -3$

10 $-(x+1)^{-3}, -(x-1)^{-3}$

11 **a** $f'(x) = 12x^2 + 5$

b $x^2 \geq 0$ for all x, so $12x^2 + 5 \geq 5$ for all x.

12 $f(x) = a - 2x$

$f'(a) = -a$

Hence $g(x) = -ax + c$.

$g(a) = 0 \Rightarrow -a \times a + c = 0, c = a^2$

13 $x^2 - 1$

14 $\frac{1-x}{\sqrt{x}}$

EXERCISE 1.2

1 D

2 B

3 $f'(x) = 84x^3 + 84x^2 - 6x - 4$

4 **a** $15x^4 + 4x^3$ **b** $24x + 1$

c $112x - 35$ **d** $7x^6 - 20x^4$

e $24x^5 - 12x^2$ **f** $50x$

g $9x^2 + 2x - 3$ **h** $32x^3 + 30x^2 - 16x - 6$

i $4x^3 + 4x$

5 -18

6 **a** $14x^6 + 44x^3 + 6x^2 + 14x$

b -38

7 14

8 **a** $y = x - 5$ **b** $\left(\frac{7}{4}, \frac{-25}{8}\right)$

9 $y = -3x - 10$

10 $\frac{81}{4}$

11 $f'(x)g(x) = 2bx(c + dx^2) = 2bcx + 2bdx^3$

$f(x)g'(x) = 2dx(a + bx^2) = 2adx + 2bdx^3$

$2bcx + 2bdx^3 = 2adx + 2bdx^3$

$2bcx = 2adx$

$bc = ad$

12 $a = 5, b = 3$

13 $-\frac{2}{3}, \frac{3}{4}$

14 **a** $2abx + a^2 + b^2$

b $a = 3, b = 2$ and $a = 2, b = 3$

15 $\left(2 - \sqrt{7}, 20 + 14\sqrt{7}\right), \left(2 + \sqrt{7}, 20 - 14\sqrt{7}\right)$

16 -1.9

17 $\frac{9}{5}, \frac{162\sqrt{5}}{125}$

18 $a = \frac{1}{2}$

EXERCISE 1.3

1 E

2 9

3 $\frac{-39}{(9x-8)^2}$

4 **a** $\frac{-4}{(2x+3)^2}$ **b** $\frac{1}{(x-5)^2}$ **c** $\frac{2}{(x+1)^2}$

d $\frac{-(2x+1)}{x^2(x+1)^2}$ **e** $\frac{2x^3 + 9x^2 + 3}{(x+3)^2}$ **f** $\frac{x^2 - 1}{x^2}$

g $2x + 1$ **h** $\frac{6x^2 + 30x + 6}{(x^2-1)^2}$

5 $\frac{1}{4}$

6 $f'(x) = -\frac{2k}{(x-k)^2}$

$f'(5) = -8 \Rightarrow \frac{2k}{(5-k)^2} = -8$

$4k^2 - 41k + 100 = 0$

$(4k - 25)(k - 4) = 0$

$k = 4$

7 $f'(x) = -\frac{1}{2(x-2)^2}$

8 -5

9 $y = x - 7, (0, -7), (7, 0)$

10 $a = 5, b = -\frac{1}{2}$

11 $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

$= -\left(\frac{1}{1+x}\right)^2$

$= -y^2$

12 $\frac{x(x+4)}{(x+2)^2}$

13 $x = -4.11, x = 0.58$

14 $-\frac{3}{4}$ at $x = 0$ and $-\frac{2}{3}$ at $x = \pm 2$

15 $a = \frac{1}{2}$

16 $a > -1$

EXERCISE 1.4

1 $\frac{2x^2-6x}{(2x-3)^2}$ **2** A

3 **a** $-\frac{24x^2}{(x^3+1)^5}$ **b** $\frac{x}{\sqrt{x^2-1}}$

4 **a** $5(x-5)^4$ **b** $16(4x-3)^3$

c $3(6x^2+1)(2x^3+x)^2$ **d** $-24x(8-2x^2)^5$

e $\frac{9}{2}\left(\frac{1}{2}x-6\right)^8$

f $2(3x^2-4x+1)(x^3-2x^2+x+1)$

g $\frac{\sqrt{2}}{\sqrt{2x+3}}$ **h** $\frac{3(1-\sqrt{x})(x-2\sqrt{x})^2}{\sqrt{x}}$

i $\frac{\sqrt{5}}{2\sqrt{x+10}}$ **j** $-\frac{4}{(2x+7)^3}$

k $\frac{1}{2\sqrt{(4-x)^3}}$ **l** $-\frac{15}{2\sqrt{(x-8)^5}}$

5 **a** $8(2x-1)^3$

b $-6x^2(3-x^3)$

c $28(1+x)(3+4x+2x^2)^6$

d $12(x+3)(x^2+6x)^5$

e $15x^2(1-2x^3)(x^3-x^6+1)^4$

f $(n+1)x^n(x^{n+1}+1)^n$

6 $5(6x-5)(3x^2-5x)^4$

7 $4(x^2-5x)^3(2x-5)$ or $\frac{dy}{dx}=4x^3(x-5)^3(2x-5)$

8 $-\frac{1}{2\sqrt{4-x}}$

9 $3(-9x^2+2x)(-3x^3+x^2-64)^2$

10 $a=1$

11 $\frac{dy}{dx}=\frac{x-a}{\sqrt{1+(x-a)^2}}$

$\sqrt{1+(x-a)^2}$ is positive for all values of x and $x-a$ will be positive if $x>a$.

12 $y=[1-f(x)]^{\frac{1}{2}}$

$y'=\frac{1}{2}[1-f(x)]^{-\frac{1}{2}}(-f'(x))$

$y'=\frac{-f'(x)}{2(1-f(x))^{\frac{1}{2}}}$

13 $f(x)=(x-a)^2g(x)$

$f'(x)=2(x-a)g(x)+g'(x)(x-a)^2$

$f'(x)=(x-a)[2g(x)+(x-a)g'(x)]$

14 $\frac{-4}{27}$ **15** $a=2, b=3$ **16** $a=\frac{9}{4}, b=1$

17 $\frac{dh}{dt}=(2t^3+2t+1)(3t^2+1)$

$t=0.1, \frac{dh}{dt}=1.2\text{ cm/m}$

EXERCISE 1.5

1 A **2** E

3 $2x(5x+1)(2x+1)^2$ **4** $u=3, v=-5$

5 $10(1-p)^8(1-10p)$

6 **a, b** $\frac{-3}{(2x-1)^2}$

7 $a=3$ **8** Proof: see worked solutions

9 0.5 **10** $a=1, b=3$

11 Proof: see worked solutions

12 **a** $f'(x)=2(2x-1)(x^2-x+1), g'(x)=3(x+a)^2$

b $f'(0)=-2$

$g'(0)=3a^2$

$f'(0)\times g'(0)=-1 \Rightarrow -2\times 3a^2=-1$ or $6a^2=1$

13 Proof: see worked solutions

14 **a** $a=1, b=12$ **b** $(3.231, 0.566)$

15 $a=9, b=5$

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 $35(5x+1)^6$ **2** $30x^2(2x^3+1)^4$

3 **a** $\frac{2}{(x+2)^2}$ **b** -9

4 $-\frac{1}{\sqrt{1-2x}}$

5 At $(2,4)$ $y'=4$

Substitute $(2,4)$ and $y'=4$ to obtain tangent line $y=4x-4$.

Substitute $x=3$ into tangent line and get $y=4(3)-4=8$.

6 $-\frac{3}{5}(x-4)(x-2)$

7 **a** $\frac{1}{4}$ **b** 20

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 $y=41x-31$

2 Proof: see worked solutions

3 $a=4, b=5$

4 $\left(-1,-\frac{10}{3}\right), (-3,-2)$

CHAPTER 2

EXERCISE 2.1

1 0.0063 **2** -10.05 cm^3 **3** -0.475

4 $3\frac{1}{600}$ **5** $-\frac{17}{200}$ **6** 1.6 cm^2

7 -0.002 cm **8** 0.094 cm^3

EXERCISE 2.2

1 0.08 cm

2 -15.7 cm^2

3 a $\dfrac{-1}{4(x-3)^{\frac{3}{2}}}$

b $\dfrac{24}{(x+5)^3}$

c 0

d $\dfrac{9}{8x^{\frac{1}{2}}}$

e $84x^5+30x^4+360x^3+108x^2+324x+54$

f 0

4 73

5 Concave down for $x<0.22$, concave down for $x>0.22$, point of inflection $(0.22, 0.02)$.

6 The exponential function $f(x)=e^x$ or any function that equals a constant, e.g. $f(x)=6$.

7 1

8 $a=1$ and $b=3$

9 $x=1$ or $x=2$

EXERCISE 2.3

1 $\dfrac{2}{27}$

2 approx. $(-0.6, 10.6)$

3 $(2,-4)$, local minimum

4 $(1,0)$

5 $\left(-1,\dfrac{5}{2}\right)$, local maximum

6 $\left(-\dfrac{5}{4},\dfrac{2187}{512}\right)$ and $(1,0)$

7 $(-3,10)$, local minimum and $\left(1,-\dfrac{2}{3}\right)$, local minimum

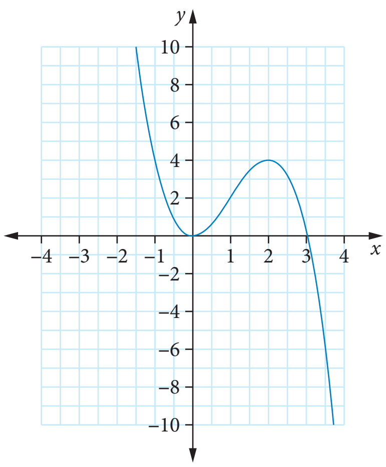
8 There is a stationary point of inflection.

9 There is a stationary point of inflection.

10 2

11 a Local minimum at $(0,0)$ and local maximum at $(2,4)$.

b



12 a F

b F

c T

d T

e F

13 $a=3, b=-9$

14 a $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{x}}-2x+1$

b Substitute $x=1$ into $\dfrac{dy}{dx}$ to get

$\dfrac{1}{1}-2(1)+1=1-2+1=0.$

Thus, $x=1$ must be a stationary point as $\dfrac{dy}{dx}=0$.

c $\dfrac{d^2y}{dx^2}=-\dfrac{1}{2x^{\frac{3}{2}}}-2$, substitute in $x=1$, $\dfrac{d^2y}{dx^2}<0$,

so a local maximum.

15 $f'(x)=\dfrac{1}{2}x^{-\frac{1}{2}}+2x$

$=\dfrac{1}{2\sqrt{x}}+2x$

Let $f'(x)=0$ to determine the stationary points.

$\dfrac{1}{2\sqrt{x}}+2x=0$

$x^{\frac{3}{2}}=-\dfrac{1}{4}$

As there is no solution for $f'(x)=0$, there are no stationary points.

16 $f'(x)=3ax^2-2bx+c=0$

$x=\dfrac{b\pm\sqrt{b^2-3ac}}{3a}$

For no solution, $b^2-3ac<0$.

Therefore, $c>\dfrac{b^2}{3a}$.

17 a $12x^2+5$

b $x^2\geq 0$, for all values of x, hence $12x^2+5\geq 5$ for all x.

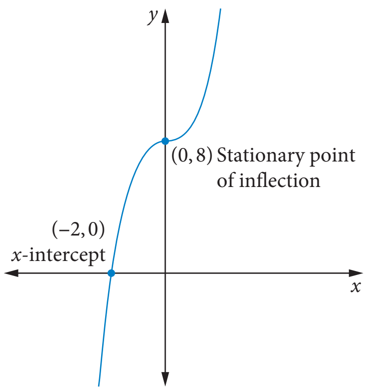
18 $m=0$, 1 or 2

EXERCISE 2.4

1 B

2 $(0,-1)$

3



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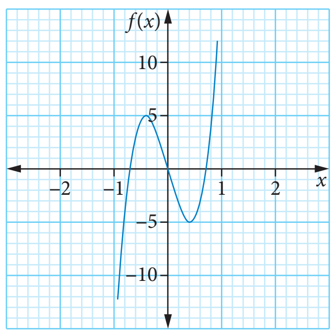
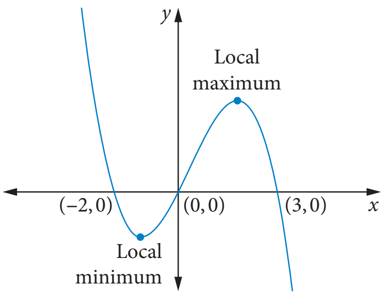


5 a



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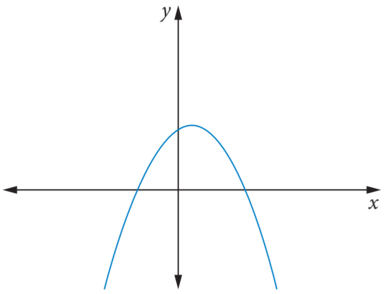

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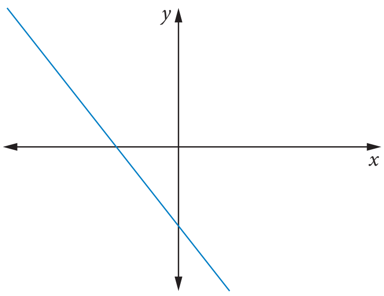

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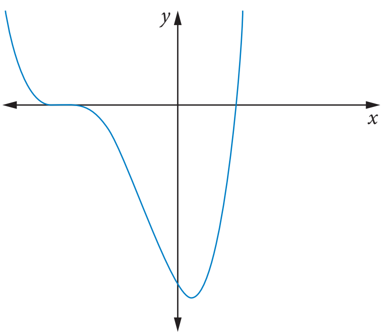


8 a $-3 < x < 3$ **b** $(3, -4)$

EXERCISE 2.5

1 E **2** B

3 a $v(t) = 4t$ **b** $4\,\text{m/s}^2$

4 a $3\,\text{m}, -0.5\,\text{m/s}, -4\,\text{m/s}^2$

b The ball is moving downwards with a speed of 0.5 m/s.

5 a $v(t) = 6t + 2$ **b** 6

6 a $v = 3t^2 + 12t - 2,\ a = 6t + 12$

b $x(5) = 266\,\text{m}$ **c** $v(5) = 133\,\text{m/s}$

d $a(0) = 12\,\text{m/s}^2$ **e** $a(5) = 42\,\text{m/s}^2$

7 a 4 m/s **b** −2 m/s **c** 14 m/s

d 6 m/s. Part **a** is the average rate of change of

displacement = average velocity $= \dfrac{x(4) - x(2)}{4 - 2} = 4\,\text{m/s}$.

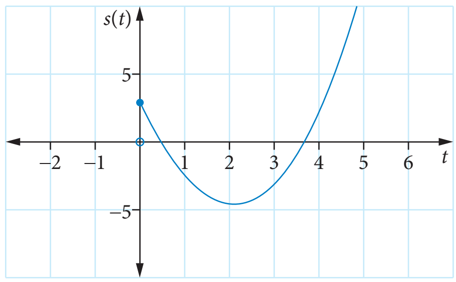
e $a = 2\,\text{m/s}^2$ **f** $a = 20\,\text{m/s}^2$

8 a $v(0) = -8\,\text{m/s}$ **b** $a(t) = 4\,\text{m/s}^2$

c $s(5) = 13\,\text{m}$ **d** $t = 2\,\text{s}$

e $s(2) = -5\,\text{m}$

f



9 a $v(5) = 120\,\text{km/h}$ **b** $a(5) = 20\,\text{km/h}^2$

10 $a(5) = 20\,\text{m/s}^2$ **11** $4\,\text{m/s}^2$

12 a $v(t) = -\dfrac{1}{(2t+5)^2}$

b $a(t) = -\dfrac{4}{(2t+5)^3}$

c $a(t) = -4 \times \dfrac{1}{(2t+5)^2} = -4 \times v(t)$

magnitude of $a(t) = 4 \times$ magnitude of $v(t)$

13 $p = -2,\ q = -12$ and $r = 20$

EXERCISE 2.6

1 −0.55 **2** 1 **3** $1.24\,\text{cm}^3$

4 0.10 units **5** $16\,\text{cm}^3$

6 $P = 8x + 4h$

$\therefore h = \dfrac{P}{4} - 2x$

$V = x^2 h$

$V = x^2\left(\dfrac{P}{4} - 2x\right)$

$\dfrac{dV}{dx} = \dfrac{Px}{2} - 6x^2$

$\dfrac{Px}{2} - 6x^2 = 0$

$x = 0, x = \dfrac{P}{12}$

Substitute $x = \dfrac{P}{12}$ into $P = 8x + 4h$ to get $h = \dfrac{P}{12}$.

Therefore, the shape must be a cube as $x = h$.

7 a $V = \dfrac{1}{4}xy(P - 4x - 4y)$

b $V = \dfrac{1}{2}x^2(P - 12x)$

$\dfrac{dv}{dx} = 0 \Rightarrow x(P - 18x) = 0 \Rightarrow x = \dfrac{P}{18}$

$V = \dfrac{1}{2}\left(\dfrac{P}{18}\right)^2\left(P - 12 \times \dfrac{P}{18}\right) = \dfrac{P^3}{6 \times 18^2}\,\text{cm}^3$

8 **a** $S = x^2 + y^2 \Rightarrow y = \sqrt{S - x^2}$

product: $p = xy = x\sqrt{S - x^2}$

b $\frac{dp}{dx} = \sqrt{S - x^2} - \frac{x^2}{\sqrt{S - x^2}}$

$\frac{dp}{dx} = 0$

$\sqrt{S - x^2} - \frac{x^2}{\sqrt{S - x^2}} = 0$

$x = \frac{\sqrt{S}}{\sqrt{2}}$

9 **a** $y = \frac{4000\sqrt{3}}{3x^2}$

b $A = 2\left(\frac{x^2}{2}.\frac{\sqrt{3}}{2}\right) + 3xy$

$= \frac{4000\sqrt{3}}{x} + \frac{x^2\sqrt{3}}{2}$

c $x = 10\sqrt[3]{4}$ (or $\sqrt[3]{4000}$)

10 1.13 cm

11 $x = 10$ cm

12 **a** $S = 10\pi r - 3\pi r^2$ **b** $r = \frac{5}{3} = 1\frac{2}{3}$ cm

13 **a** $A = 18a - 6a^3$ **b** $A = 12$ when $a = 1$

14 $P(4, 2)$, minimum length $2\sqrt{5}$

15 $3\sqrt{3}$ units^2

16 **a** $G\left(\frac{28}{9}, -\frac{50}{243}\right)$ **b** $y = -\frac{1}{8}x + \frac{1}{2}$

c $x = \frac{2\left(\sqrt{31} + 7\right)}{9}$ **d** $k = 8m$

e $x = 2^{\frac{2}{3}}$

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 $y = x + 1$

2 **a** $f'\left(\frac{5}{3}\right) = 0,\ f''\left(\frac{5}{3}\right) = 10$

b As $f'\left(\frac{5}{3}\right) = 0$, this must be a stationary point.

As $f''\left(\frac{5}{3}\right) > 0$, it must be a minimum turning point.

3 Let $f(x) = \frac{2x^2 + 1}{\sqrt{x}}$

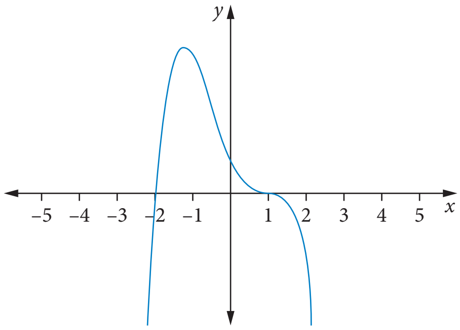
$$f'(x) = \frac{x^{\frac{1}{2}}(4x) - \left(2x^2 + 1\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}$$

$$= \frac{4x^{\frac{3}{2}} - x^{\frac{3}{2}} - \frac{1}{2x^{\frac{1}{2}}}}{x}$$

$$= \frac{3x^{\frac{3}{2}} - \frac{1}{2x^{\frac{1}{2}}}}{x}$$

$$= \frac{6x^2 - 1}{2x^{\frac{3}{2}}}$$

4



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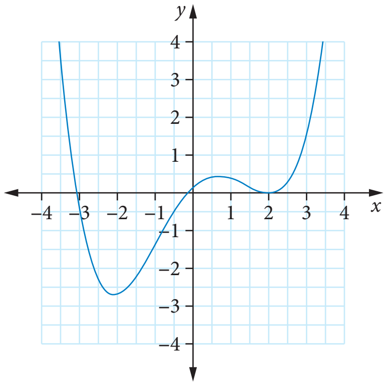


CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 The approximate change in area is 0.87 cm^2.

2 **a** $375 = \pi x^2 h$

$\therefore h = \frac{375}{\pi x^2}$

$S = 2\pi x^2 + 2\pi xh$

$= 2\pi x^2 + 2\pi x\left(\frac{375}{\pi x^2}\right)$

$= 2\pi x^2 + \frac{750}{x}$

b Cans have a radius of 3.9 cm and a height of 7.8 cm to minimise surface area.

3 $a = -18$, $b = 108$

4 **a** zero **b** 39 m/s

c $t = \frac{2}{3}$ or 2 seconds **d** -8 m/s^2

e 2.37 m

5 **a** Let the height of the tank be h.

$V = xyh = 8 \Rightarrow h = \frac{8}{xy}$

$A = xy + 2xh + 2yh$

$= xy + 2x\left(\frac{8}{xy}\right) + 2y\left(\frac{8}{xy}\right)$

$= xy + \frac{16}{x} + \frac{16}{y}$

b $\frac{dA}{dy} = x - \frac{16}{y^2} = 0$ when $y = \frac{4}{\sqrt{x}}$

$\therefore A = \frac{4x}{\sqrt{x}} + \frac{16}{x} + \frac{16\sqrt{x}}{4}$

$= 8\sqrt{x} + \frac{16}{x}$

$a = 8$, $b = 16$

CHAPTER 3

EXERCISE 3.1

1 $y=\frac{x^4}{4}+x^3-2x^2+c$

2 $\frac{1}{16}(4x-1)^4+c$

3 a $\frac{x^3}{3}-\frac{3x^2}{2}+2x+c$ b $\frac{2x^3}{3}-x^2-12x+c$

c $\frac{x^2}{2}-2x+c$ d $\frac{2x^{\frac{3}{2}}}{3}+\frac{1}{x}-3x+c$

e $\frac{(2x-3)^{\frac{3}{2}}}{3}+c$ f $\frac{2x^{\frac{7}{2}}}{7}-\frac{4x^{\frac{5}{2}}}{5}+2x^{\frac{3}{2}}+c$

g $\frac{-1}{2(2x-3)}+c$ h $\frac{(3x-4)^{\frac{4}{3}}}{4}+c$

4 $f(x)=x^2-3$

5 $y=x^2+4x-11$

6 $f(x)=-3(-2x+4)^{\frac{2}{3}}+22$

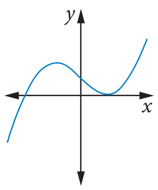
7 $y=x^3+x^4-2x+c$

8 $\frac{-1}{9(3x+4)^3}+c$

9 $\frac{1}{8}(4-2x)^{-4}+c$

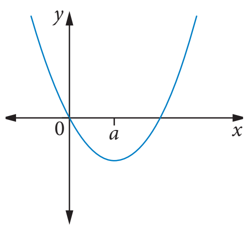
10 $f(x)=x^3-x^2-48$

11

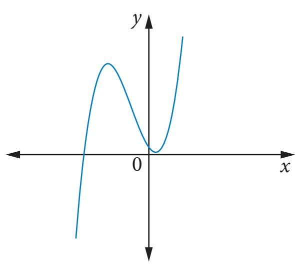


Answer can vary. This is one possible answer. Check with your teacher for other possible answers.

12 a



b



13 $2\sqrt{x^2-3x}+c$

14 $f(x)=\frac{2}{3}x^3-\frac{3}{4}x^{\frac{1}{3}}-\frac{5}{3}$

15 $\frac{-1}{4(2x-1)^2}+c$

16 $f(x)=2\sqrt{2x-3}-2$

17 $f(x)=g(x)+3x+c$

$\therefore f(0)=g(0)+3(0)+c$

$\therefore 2=1+c\Rightarrow=1$

$f(x)=g(x)+3x+1$

EXERCISE 3.2

1 $y=\frac{x^3}{3}-\frac{3x^2}{2}+c$

2 $f(x)=x^2-\frac{3}{5}x^{\frac{5}{3}}-\frac{7}{5}$

3 a 3.625 b 100 c 10 d 9.33

4 a 0.24 units^2 and 0.44 units^2

b 17.45 units^2 and 20.95 units^2

c 1.57 units^2 and 2.17 units^2

5 1.675 units^2

6 $\frac{\pi}{6}\left(1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)=\frac{\pi}{12}\left[3+\sqrt{3}\right]$

7 a 17.5 units^2 b 15.25 units^2

8 n is larger and h is smaller

9 a lower limit $= 20\times0.5+21\times0.5+24\times0.5$
$= 10+10.5+12$
$= 32.5$

upper limit $= 21\times0.5+24\times0.5+29\times0.5$
$= 10.5+12+14.5$
$= 37$

Therefore, $\int_0^{1.5} f(x)\,dx$ is between these values as this is the area under the curve.

b 75

c By reducing the width of the rectangles and, therefore, using more rectangles to estimate the area, the error in the estimate would be reduced. Another method involves determining the function and using calculus.

10 5.146

11 $f(1)+f(2)+f(3)$ because that gives the heights of the underestimated rectangles.

12 16.25 units^2

EXERCISE 3.3

1 4.75 units^2

2 It is more accurate because there are smaller spaces between rectangles and the curve.

3 a 16 b 128 c 15

d $\frac{49}{3}$ e $\frac{1}{3}$ f $\frac{1}{4}$

4 a $\frac{40}{3}$ b $-\frac{40}{3}$

5 a $2x^2+x-4$ b $\frac{3}{x^2-1}$

6 -15

7 7

8 **a** $\frac{4}{3}$ **b** 2 **c** 6 **d** 50 **e** $\frac{23}{6}$

9 **a** $\sqrt{x-\pi}$ **b** $2x^2 - x$

10 $\frac{9x^2}{2} + 2(1-2x^2)$

11 $\frac{23}{3}$

12 **a** $\int_0^5 x^2 dx$ **b** $\int_1^7 (x+1)\,dx$ **c** $\int_{-2}^{2} (x^3 - x - 1)\,dx$ **d** $\int_0^3 (2x+1)\,dx$

13 $\frac{1}{12}$

14 9 **15** 5 **16** 4 **17** -1

18 $\int_4^8 f(x)\,dx = F(8) - F(4) = F(8) + 6$

$\Rightarrow F(8) = -6 + \int_4^8 f(x)\,dx$

19 25

EXERCISE 3.4

1 $\frac{3}{4}$ **2** 4

3 **a** 7.5 units^2 **b** 39 units^2 **c** 74 units^2

4 $24\frac{3}{4}\text{ units}^2$ **5** $78\frac{1}{12}\text{ units}^2$

6 **a** 9 **b** 19

7 $\int_{-2}^{-1} f(x)\,dx$ **8** $\int_{-3}^{0} f(x)\,dx - \int_0^1 f(x)\,dx$

9 $\frac{4}{3}\text{ units}^2$ **10** $\frac{4}{15}\text{ units}^2$ **11** 9.5

12 $-\int_{-1}^{1} f(x)\,dx + \int_1^4 f(x)\,dx - \int_4^6 f(x)\,dx$

13 $2\sqrt{3}$ **14** 40.5 units^2

EXERCISE 3.5

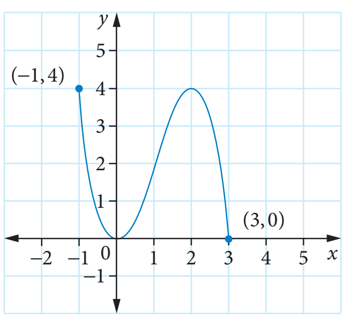
1 D **2** 0.5 units^2 **3** $\frac{125}{6}\text{ units}^2$

4 **a** $\frac{32}{3}\text{ units}^2$ **b** $\frac{32}{3}\text{ units}^2$ **c** $\frac{125}{24}\text{ units}^2$

5 **a** $\frac{4}{3}\text{ units}^2$ **b** $\frac{500}{3}\text{ units}^2$ **c** 9 units^2

6 **a** $(0,0)$ and $(2,4)$

b



c 6.75 units^2

7 $\int_{-3}^{2}[g(x) - f(x)]\,dx$

8 $\int_{-1}^{2} f(x)\,dx - \int_{-1}^{2} g(x)\,dx$

9 $\int_{-1}^{1}[f(x) - g(x)]\,dx + \int_1^4 [g(x) - f(x)]\,dx$

10 4.5 units^2

11 **a** $\frac{32}{3}\text{ units}^2$ **b** $\frac{49}{2}\text{ units}^2$ **c** $\frac{22}{3}\text{ units}^2$ **d** 36 units^2

12 $\frac{125}{6}\text{ units}^2$ **13** 3.083 units^2

14 $\frac{8}{3}\text{ units}^2$ **15** $a = 8$, $m = 4$

EXERCISE 3.6

1 E **2** B **3** $v(t) = 2t^2 + t$

4 **a** $v(t) = 3t - t^2 + 2$

b $\frac{25}{6}$ m. The particle changes its position by $\frac{25}{6}$ m between $t = 1$ and $t = 2$ seconds.

c Since the particle's velocity is positive between $t = 1$ and $t = 2$, the distance travelled is $\frac{25}{6}$ m.

5 $x(t) = -\frac{t^3}{3} + \frac{t^2}{2} + \frac{2}{3}$ **6** $x(t) = t^4 + t^3$

7 $v(t) = 2t^2 - 8$ **8** $x(t) = \frac{3t^2}{2} - \frac{2t^3}{3}$

9 **a** $v(t) = 8 + 0.4t$

b $x(t) = 8t + 0.2t^2$

c 100 m

10 81.7 m

11 **a** $k = 1.5$

b 17 min

c 1125 m below the mountain station

12 **a** 4 m/s **b** $\frac{128}{3}$ m

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 $\sqrt{8} = 2\sqrt{2}$

2 $\frac{5}{2}$

3 $f(x) = \frac{1}{3}x^2 - 5x^2 - 2\sqrt{x} + x + 80$

4 $a = 2, b = 0, c = -6, d = 4$

5 **a** $46 + 4\pi$ **b** $20 + 8\pi$ **c** $\frac{20 + 8\pi}{6} + 18$

6 **a** $\frac{4}{3}\text{ units}^2$

b Both graphs from part **a** have been vertically translated down by 5 units. The shape of both graphs is unchanged. Therefore, the area between them remains unchanged.

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 -16 **2** $\frac{4}{3}$ units^2

3 a -13 cm/s^2 **b** 12

c $\frac{1}{2}, \frac{5}{3}$ seconds **d** 115.7 cm

4 a i 3 **ii** -2

b 8 units^2

5 a $\left(\frac{2a}{3}, \frac{3}{a}\right)$ **b** $0, \frac{a}{3}, \frac{2a}{3}$ **c** $\frac{1}{8}$ units^2

CHAPTER 4

EXERCISE 4.1

1 a i $N_0 = 200$ **ii** $k = 0.9163$

b 48 828 people

2 a i $B_0 = 100\,000$ **ii** $k = 0.0098$

b 126 389 **c** 71 h

3 a $D_0 = 400$ **b** $k = 0.3567\ldots$ **c** 816.4 mg

4 a 6191 **b** 6439 **c** 4%

5 a $10 = d_0e^m$, $15 = d_0e^{2m}$ **b** $d_0 = 6.667$, $m = 0.405$

6 a $S_0 = 163$ **b** $k = 0.5597$ **b** 76 914

7 a $P_0 = 200$ **b**
$$\begin{aligned} 100 &= 200e^{5k} \\ \frac{1}{2} &= e^{5k} \\ 2 &= \frac{1}{e^{5k}} \\ e^{-5k} &= 2 \end{aligned}$$

8 a $k = -0.006\,90 = -6.90 \times 10^{-3}$ **b** 434.42 days

9 a 4 mg/L **b** 3.53 mg/L

10 a $k = 0.0693$ **b** 1.25×10^{19}

c The graph has a horizontal asymptote at $N = 0$, therefore, it will never reach zero where there is no sample left.

EXERCISE 4.2

1 a 50 **b** 1.0986

2 0.0347

3

a	$\lim\limits_{h\to 0} \frac{a^h - 1}{h}$
2.71	0.996 949
2.711	0.997 318
2.712	0.997 686
2.713	0.998 055
2.714	0.998 424
2.715	0.998 792
2.716	0.999 160
2.717	0.999 528
2.718	0.999 896
2.719	1.000 264
2.72	1.000 632

The best approximation is 2.718.

4 a $9e^x$ **b** $e^x + 2x$

c $12e^x(2e^x - 3)^5$ **d** $e^{-3x}(e^{4x} - 2e^{2x} - 3)$

e $2e^{2x-1}$ **f** $\dfrac{e^{\sqrt{2x+4}}}{\sqrt{2x+4}}$

5 a $xe^x + e^x$ **b** $2xe^x + 5e^x$ **c** $5x^3e^x + 15x^2e^x$

6 $\dfrac{e^3(e^3 + 6)}{2(e^3 + 1)^{\frac{3}{2}}}$

7 a $\dfrac{(x-2)e^x}{x^3}$ **b** $\dfrac{(6x-1)e^{6x}}{3x^2}$ **c** $\dfrac{2(5x-3)e^{5x}}{5x^4}$

d $\dfrac{2-x}{e^x}$ **e** $-\dfrac{e^x + 2}{e^{2x}}$

8 $\dfrac{28}{e^4}$ **9** 348.4

10 $80e^6 + e^2$ **11** $x = 2$

12 $y = \dfrac{\sqrt{e}}{2}(x+1)$ **13** $y = \dfrac{e^3}{2}(3x - 4)$

14 a $(4x-4)e^{2x^2-4x}$ **b** $(16x^2 - 32x + 20)e^{2x^2-4x}$

c $x = 1$ **d** local minimum

15 a 1100 **b** 1246

c 31 swans per month **d** 3.1

16 a 3707 hectares per year

b 2414 hectares per year

c 1058 hectares per year

17 a 200 mg **b** $k = -0.043$

c -6.355 mg/day

18 a $3x^2e^{2x} + 2x^3e^{2x}$ **b** $6e^9$ **c** e^3

19 a Proof: see worked solutions

b $f'(x) = \dfrac{-(x-1)(x-3)}{e^x}$

$f'(1) = 0 = f'(3)$

20 $k = e^a(a - 1)$

21 a 25°C **b** 76.68°C

c 223°C **d** 8.63°C/min

e As time increases, the rate of change in the temperature of the water → 0.

The temperature of the water → the constant value of T_0.

22 a

h	$a = 2.60$	$a = 2.70$	$a = 2.72$	$a = 2.80$
0.1	1.002 65	**1.044 25**	1.052 41	1.084 49
0.001	0.955 97	0.993 75	**1.001 13**	**1.030 15**
0.000 01	0.955 52	**0.993 26**	**1.000 64**	1.029 62

b $a = e \approx 2.71828$

When $a = e$, the table shows that the value of

$\lim\limits_{h\to 0}\left(\dfrac{a^h - 1}{h}\right)$ is 1.

It follows then from the definition that

$$\begin{aligned} \frac{d}{dx}(e^x) &= e^x \times 1 \\ &= e^x \end{aligned}$$

EXERCISE 4.3

1 218.393

2 $3e^6$

3 a $-\frac{1}{2}e^{-2x}+c$ **b** $\frac{5}{4}e^{4x}+c$

c $\frac{1}{2}e^{2x+1}+c$ **d** $-\frac{3}{2}e^{-2x}+\frac{1}{4}e^{4x}+c$

e $\frac{1}{3}e^{3x}+e^{-x}+c$ **f** $\frac{1}{6}e^{6x}+\frac{1}{6}e^{-6x}-2x+c$

4 $y=\frac{1}{2}e^{4x}+\frac{3}{2}$

5 $y=\frac{1}{2}e^{2x}+e^{-x}+\frac{19}{2}$

6 $f(x)=-\frac{5}{2}e^{-2x}+\frac{4e^7+5}{2e^6}$

7 a e^4-1 **b** $5e^3-5e$ **c** $-e^4+e^2+60$

8 a $\frac{e^{\pi}}{6}-\frac{e^{-\pi}}{6}\approx 3.85$ **b** $\frac{e^{3\pi}}{3}+\frac{5}{3}\approx 4132.22$

9 -5306.0

10 $\frac{2e^3}{3}+\frac{1}{3}$ unit^2

11 a $2(2x+1)e^{2x}$ **b** $(2x-1)e^{2x}+c$

12 $2\left(2+\frac{1}{e^4}\right)$ units^2

13 a $y=-2x+2$ **b** $2e^{\frac{1}{2}}-2$ units^2

14 a $xe^{-\frac{x}{5}}$ units^2 **b** $\frac{5}{e}$ at $x=5$ units^2 **c** 1.16 units^2

15 a 0.9975 mm^3/mm

b It is the rate of change of the volume with respect to height when the height has reached 0.5 m.

c i 5 m/s **ii** 3.488 m^3/s **iii** 8.594 m^3

EXERCISE 4.4

1 $\frac{e^{4x}}{4}-2x-\frac{e^{-4x}}{4}+c$

2 $2e^6-2$

3 $2x\sin(x)+x^2\cos(x)$

4 $\frac{dy}{dx}=-\frac{10\cos(3x)\cos(2x)+15\sin(3x)\sin(2x)}{\sin^2(2x)}$

5 $\frac{dy}{dx}=-(20x^4-1)\sin(4x^5-x)$

6 $\frac{dy}{dx}=-12x^4\sin(x^3)+8x\cos(x^3)$

7 a -0.269 **b** 1.047

8 a $v=-\frac{3}{2}\sin\left(\frac{t}{2}\right)$ **b** $a=-\frac{3}{4}\cos\left(\frac{t}{2}\right)$

c π s

d $v=-\frac{3}{2}$ m/s, $a=0$ m/s^2 **e** 2π s

9 a Proof: see worked solutions

b Proof: see worked solutions **c** $(1.9, 2.4)$

10 a $(4.31, 3.78)$

b maximum at $(5.94, 0.67)$, minimum at $(2.08, -1.74)$

11 $h'(x)=\frac{e^{-x}\sin(x)-e^{-x}\cos(x)}{\cos^2(x)}$, $h'(\pi)=e^{-\pi}$

12 a $f'(x)=\sin(x)+x\cos(x)$ **b** $f'\left(\frac{\pi}{2}\right)=1$

13 $\frac{\pi\times\sqrt{3}}{6}+\frac{\pi^2}{36}$

14 a $P=\left(\frac{\pi}{6},4\right)$ **b** $y=-6x+\pi+4$

15 a 0 micrometres/s

b 45 micrometres/s^2

c -15 micrometres/s

16 -4 m/s^2

17 a

$$\begin{aligned}\tan\theta &= \frac{y}{12}\\ y &= 12\tan\theta\\ &= \frac{12\sin\theta}{\cos\theta}\\ \frac{dy}{d\theta} &= \frac{12\cos\theta\cos\theta+12\sin\theta\sin\theta}{\cos^2\theta}\\ &= \frac{12}{\cos^2\theta}\end{aligned}$$

b 265.465 km/min

18 $y=\frac{\sin(x)}{\cos(x)}$

$u=\sin(x)$ $\quad v=\cos(x)$

$\frac{du}{dx}=\cos(x)$ $\quad \frac{dv}{dx}=-\sin(x)$

$$\frac{dy}{dx}=\frac{\cos(x)\times\cos(x)+\sin(x)\times\sin(x)}{(\cos(x))^2}$$

$$\frac{dy}{dx}=\frac{\cos^2(x)+\sin^2(x)}{(\cos(x))^2}=\frac{1}{\cos^2(x)}$$

19 a $-\frac{1}{2}$

b

$$\begin{aligned}y-\frac{\sqrt{3}}{2} &= -\frac{1}{2}\left(x-\frac{2\pi}{3}\right)\\ y &= -\frac{1}{2}x+\frac{\pi}{3}=\frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\text{At } (c,0)\text{: } 0 &= -\frac{1}{2}c+\frac{\pi}{3}+\frac{\sqrt{3}}{2}\\ c &= \sqrt{3}+\frac{2\pi}{3}\end{aligned}$$

EXERCISE 4.5

1 12 m/s^2

2 $\frac{\pi}{2}$

3 a $-3\cos(2x)+c$ **b** $\sin\left(\frac{x}{2}\right)+c$

c $-\frac{6}{5}\cos(5x-7)+c$ **d** $\frac{1}{2}$

e $\sqrt{3}-1$ **f** $\frac{1-\sqrt{2}}{3}$

4 $6x\sin(2x)+3\cos(2x)+c$

5 $f(x)=\sin(x)-\frac{1}{3}\cos(3x)+2$

6 8 units^2

ANSWERS

7 **a** $a(t) = \frac{7\pi^2}{320}\cos\left(\frac{\pi t}{40}\right)\text{m/s}^2$

b $\frac{7\pi}{16}$

c $x(t) = 35 - 35\cos\left(\frac{\pi t}{40}\right)$

8 2.75 units^2

9 **a** $\frac{1}{2}\sin(2x+1)+c$

b $f(x) = 2\sin(x) + \frac{1}{2}\cos(2x) - 1$

10 $\int x\sin(3x)\,dx = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} + c$

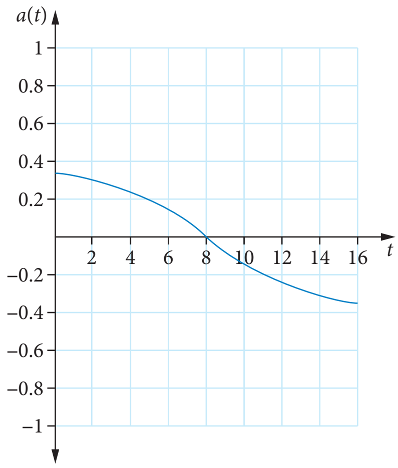
11 $s(t) = -2\sin(t) + 2\cos(3t) - 2t + 2$

12 $s(t) = -\cos(t) + t + 2$

13 **a** $2\sin(3x) + 6x\cos(3x)$

b
$$\frac{d(2x\sin(3x))}{dx} = 2\sin(3x) + 6x\cos(3x)$$
$$\int \frac{d(2x\sin(3x))}{dx}\,dx = \int (2\sin(3x) + 6x\cos(3x))\,dx$$
$$2x\sin(3x) + c_1 = \int 2\sin(3x)\,dx + 6\int x\cos(3x)\,dx$$
$$\frac{2x\sin(3x)+c_1}{6} = \frac{-2\cos(3x)}{18} + c_2 + \int x\cos(3x)\,dx$$
$$\int x\cos(3x)\,dx = \frac{2x\sin(3x)}{6} + \frac{2\cos(3x)}{18} + c$$
$$\therefore \int x\cos(3x)\,dx = \frac{3x\sin(3x)+\cos(3x)}{9} + c$$

14 **a**



$a(t) = \frac{9\pi^2}{256}\cos\left(\frac{\pi t}{16}\right)\text{m/s}^2$

b Since the acceleration is positive in the interval $0 < t < 8$, the velocity is increasing in the interval $0 < t < 8$.

Since the acceleration is negative in the interval $8 < t < 16$, the velocity is decreasing in the interval $8 < t < 16$.

c $x(t) = 9 - 9\cos\left(\frac{\pi t}{16}\right)$ $\quad x(16) = 18\text{ m}$

15 **a**
$$A(p,q) = \int_0^q \left(10\sin\left(\frac{x}{15}\right) + p\right)dx$$
$$= pq - 150\cos\left(\frac{q}{15}\right) + 150$$
$$p + q = 500$$
$$\therefore\ p = 500 - q$$
$$A(q) = q(500-q) - 150\cos\left(\frac{q}{15}\right) + 150$$
$$= 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$

b $q \approx 247$, $62\,750\text{ m}^2$

16 **a** 864 cm

b $h'(x) = 4\cos\left(x - \frac{3\pi}{2}\right) - 2x + 3\pi$

c $x = 5.74$ m. Hence, the maximum height $h(5.74) = 20.57$ m.

17 $\frac{7}{9}$

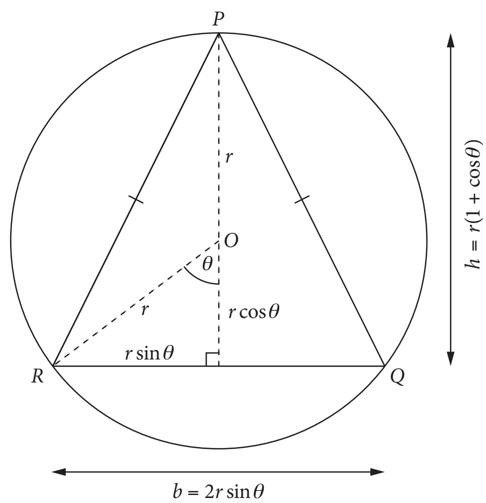
CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 **a** gradient = 15 **b** minimum turning point

2 **a** $30x^2(2x^3+1)^4$ **b** $-2e^{\pi}$

c $\int 3\cos(2x)\,dx = \frac{3}{2}\sin(2x) + C$

3 **a** Use $A = \frac{1}{2}bh$ with the following measurements:



b
$$A = r^2\sin\theta(1+\cos\theta)$$
$$\frac{dA}{d\theta} = r^2\left[\sin\theta(-\sin\theta) + (1+\cos\theta)\cos\theta\right]$$
$$= r^2\left[\cos\theta + \cos^2\theta - \sin^2\theta\right]$$
$$= r^2\left[\cos\theta + \cos^2\theta - (1-\cos^2\theta)\right]$$
$$= r^2\left[2\cos^2\theta + \cos\theta - 1\right]$$
$$= r^2(2\cos\theta - 1)(\cos\theta + 1)$$

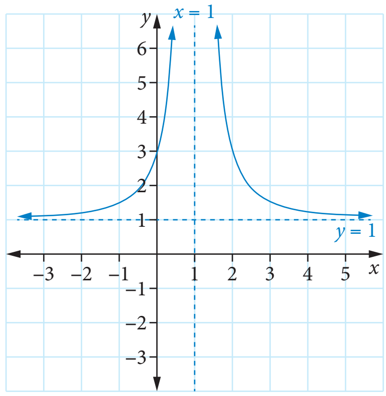
$$\frac{dA}{d\theta} = 0 \quad \cos\theta = \frac{1}{2}, \theta = \frac{\pi}{3}, \cos\theta \neq -1, 0 < \theta < \pi$$

$$A = r^2\sin\theta(1+\cos\theta)$$
$$= r^2\frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)$$
$$= \frac{3\sqrt{3}}{4}r^2$$

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 a i $\frac{3}{2}$

ii



b 2 units^2

2 a 2800

b 176 animals/year

c $t = 14.8$, October 2031

3 a $x(t) = -6\cos\left(\frac{t}{3} + \frac{\pi}{6}\right) + 3\sqrt{3}$

b The drone is 0.27 m (27 cm) due south of the pilot.

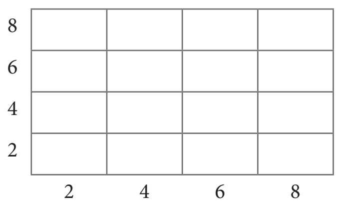
c The drone has travelled 12.696 metres.

CHAPTER 5

EXERCISE 5.1

1 $\frac{37}{145}$ **2** $\frac{10}{323}$

3 a



b i $\frac{1}{4}$ **ii** $\frac{1}{4}$

4 0.42 **5** $P(A) = 0.15$ **6** $\frac{1}{5}$

7 a $\frac{3}{4}$ **b** $\frac{2}{3}$

8 a $\frac{4}{9}$ **b** $\frac{3}{4}$

9 a $\frac{5}{6}$

b i $\frac{19}{32}$ **ii** $\frac{15}{19}$

10 a $\frac{1}{15}$ **b** $\frac{1}{3}$

11 a $\frac{2}{5}$ **b** $1 - \frac{g}{6}$

12

Score	Combinations	Probability
7	3, 4 or 2, 5 or 1, 6	$\frac{2}{8} \times \frac{1}{6} + \frac{2}{8} \times \frac{1}{6} + \frac{4}{8} \times \frac{1}{6} = \frac{8}{48}$
8	3, 5 or 2, 6	$\frac{4}{48}$
9	3, 6	$\frac{2}{48}$

probability of a prize $= \frac{8}{48} + \frac{4}{48} + \frac{2}{48} = \frac{14}{48} = \frac{7}{24}$

13 a $\frac{15}{28}$ **b** $\frac{13}{28}$

14 a 0.49 **b** 0.33

EXERCISE 5.2

1 B

2 a $\frac{4}{49}$ **b** $\frac{43}{49}$

3 a uniform discrete probability distribution

b

x	1	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

4 X represents the number of tails.

x	0	1	2
$P(X = x)$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

5

x	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

6 $p = \frac{3}{5} = 0.6$ **7** $k = \frac{1}{10}$ **8** 0.29

9 Adding all probabilities give $0.6p^2 - p + 0.4 = 0$, giving $(3p - 2)(p - 1) = 0$, so $\frac{2}{3}$ or $p = 1$.

10 $p = \frac{1}{2}$ **11** 0.3

12 a 0.008 **b** $\frac{29}{64}$

13 a

		Roll two			
	Sum of two rolls	1	2	3	4
Roll one	1	1 + 1 = 2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

b i

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

ii $\frac{10}{16} = \frac{5}{8}$

EXERCISE 5.3

1 C **2** B

3 $E(X) = 2.85$ **4** $a = 0.3, b = 0.1$

5 **a** $E(X) = 2.8$

b $Var(X) = 9 - 2.8^2 = 1.16$

c $SD(X) = 1.077$

6 $Var(X) = 1$

7 **a** 27 **b** 48 **c** $a = 2, b = 70$

8 $E(X) = 14.30, Var(X) = 26.51, SD(X) = 5.15$

9 1.5

10 **a** $\frac{28}{15}$ **b** $\frac{8}{15}$

11 **a**

n	0	1	2	3
$P(N = n)$	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

b 1.3

12 **a** 80

b 1936

c $a = \frac{15}{22} \approx 0.682, b \approx \frac{195}{22} = 8.864$

13 **a** 0.12

b 0.729

c **i** $p^2 + (1 - p)(p - 0.2) = 0.7$ gives $p = 0.75$

ii 1.21

EXERCISE 5.4

1 B **2** E

3 **a** Bernoulli distribution, $X \sim \text{Bern}(0.05)$

b mean = 0.05, variance = 0.0475

4 **a** 0.216 **b** 0.177 **c** 0.885

5 **a** 0.046 **b** 0.8122

6 **a** **i** $6p^5(1 - p)$ **ii** p^6

b $p = \frac{6}{7}$

7 **a** $n = 150, p = 0.6$ **b** 0.07

8 **a** $n = 13$ **b** $n = 11$

9 **a**

y	0	1
$P(Y = y)$	$\frac{1}{5}$	$\frac{4}{5}$

b It is a Bernoulli distribution.

c $\mu = \frac{4}{5}, \sigma = \frac{2}{5}$ **d** $X \sim \text{Bin}\left(5, \frac{4}{5}\right)$ **e** $\frac{32}{625}$

10 **a** $\frac{16}{81}$ **b** $\frac{65}{81}$ **c** $\left(\frac{2}{3}\right)^{24}$

11 **a** **i** p^3 **ii** $3p^2(1 - p)$

b $p = 0, p = \frac{3}{4}$

12 $\frac{5}{16}$

13 **a** $p = 0.2$ **b** $E(X) = 0.6$

c $Var(X) = 0.84$ **d** 0.000 45

14 **a** $X \sim \text{Bin}(5, 0.05)$

b 1 The alarms fail independently of each other.

2 The probability that an alarm fails is constant/unchanging/same for all alarms.

c 0.000 03 **d** 0.999 52

15 **a** 0.9153 **b** 0.086

16 **a** 0.9015 **b** 0.9311

17 **a** 0.1147 **b** 23

18 **a** $X \sim \text{Bin}\left(30, \frac{4}{5}\right)$

$u = 24, \sigma = \sqrt{30\frac{4}{5}\left(1 - \frac{4}{5}\right)} = 2.191$

b 0.123

19 18

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 $\frac{d^2y}{dx^2} = 12x^2 + 12$ **2** $P\left(\frac{8}{5}, \frac{4}{5}\right)$

3 **a** $\frac{125}{512}$

b

x	$P(X = x)$
0	$\left(\frac{5}{8}\right)^3$ or $\frac{125}{512}$
1	$\binom{3}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)$ or $\frac{225}{512}$
2	$\binom{3}{2}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^2$ or $\frac{135}{512}$
3	$\left(\frac{3}{8}\right)^3$ or $\frac{27}{512}$

c $\mu = \frac{9}{8}, \sigma = \frac{45}{64}$

4 **a** $\frac{1}{10}$ **b** $m = 19n - 20$

5 **a** $\frac{297}{625}$ **b** $\frac{6^3}{5^4 - 2^4}$

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 **a** $P'(t) = 2\sin(3t) + 6t\cos(3t)$\$/year

b $6\sqrt{3} - \frac{\pi}{2}$\$/year^2

c The approximate change in profit is $-\frac{1}{6}$ million dollars $\left(\frac{1}{6}\text{ million dollar loss}\right)$.

2 **a**

Amount won	20	10	0
Probability	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{20}{36}$

b \$6.11

c Liu Yang is better off in the long term. In the long term, Liu Yang will likely win \$1.11 per game.

d \$7.64

9780170477536

3 **a** $X \sim \text{Bin}(5, 0.25)$ **b** $\frac{1}{64}$

c If the carnival organisers only charge \$2 per game, then, on average, they will lose approximately 34 c per game.

4 **a** **i** 0.310

ii No, the events are not independent. $P(H|S)$ is not equal to $P(H)$.

b 0.190 **c** 0.0227

CHAPTER 6

EXERCISE 6.1

1 **a** $\log_7(49) = 2$ **b** $\log_3(27) = 3$

c $\log_2(16) = 4$ **d** $\log_5(125) = 3$

e $\log_{11}(1) = 0$ **f** $\log_2(1) = 0$

g $\log_5\left(\frac{1}{25}\right) = -2$ **h** $\log_4\left(\frac{1}{16}\right) = -2$

2 **a** $5^2 = 25$ **b** $4^2 = 16$ **c** $5^3 = 125$ **d** $2^4 = 16$

e $3^1 = 3$ **f** $7^2 = 49$ **g** $2^7 = 128$ **h** $5^0 = 1$

3 **a** 6 **b** 2 **c** 4 **d** 3

e 3 **f** 0 **g** 1 **h** 5

i 5 **j** 5 **k** 4 **l** −3

4 **a** 1 **b** 1 **c** $-\frac{1}{3}$ **d** 9

e 3 **f** 0 **g** 1 **h** 2

5 **a** $\log_4(x^4)$ **b** $\log_7(x^2)$ **c** $\log_6\left(\frac{1}{x}\right)$

d $\log_2\left(\frac{1}{x+2}\right)$ **e** $\log_4(x-1)$ **f** $\log_3(1) = 0$

6 **a** $\frac{1}{2}$ **b** $\frac{3}{2}$ **c** $\frac{3}{2}$

d $\frac{3}{2}$ **e** $\frac{2}{3}$

EXERCISE 6.2

1 **a** 5 **b** 3 **c** −4

2 **a** 5 **b** 4

3 **a** $\log_3(7) + 5$ **b** $\log_2(12) - 3$

c $\frac{1}{3}\ln(9)$ **d** $\frac{1}{2}(\ln(2) - 3)$

4 **a** $x = \ln(8)$ **b** $x = \log_5(4)$

c $x = \frac{1}{3}\log_2(3)$ **d** $x = 0, x = \frac{1}{2}\log_3(2)$

5 **a** $\frac{13}{2}$ **b** $\frac{7}{9}$ **c** $\frac{5}{6}$

d 2.2 **e** 7 **f** 22

6 **a** 4 **b** 7 **c** $\frac{8}{17}$

7 **a** 145 **b** $y = \frac{26x}{51}$

8 **a** $a = 2, b = 10$ **b** $a = 5, b = 21$

9 $x = \frac{\ln(9)}{2}$

10 **a** 32 **b** $y = -\frac{1}{2}x$

11 **a** $x = \frac{18}{25}$ **b** $x = \frac{e^2 - 10}{6}$ **c** $x = \frac{10}{3}$

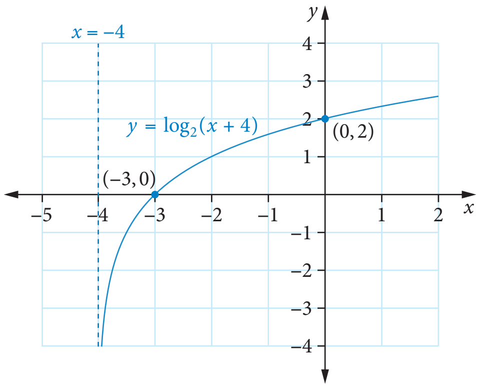
12 **a** Proof: see worked solutions

b $a = 53, b = \log_3(2)$

EXERCISE 6.3

1 17 **2** $\ln(4)$

3



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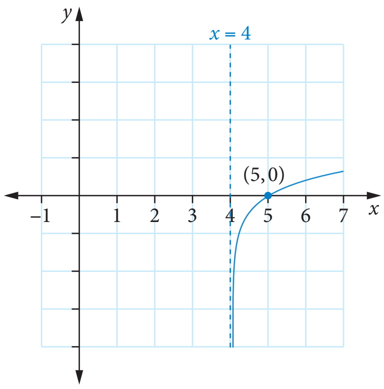
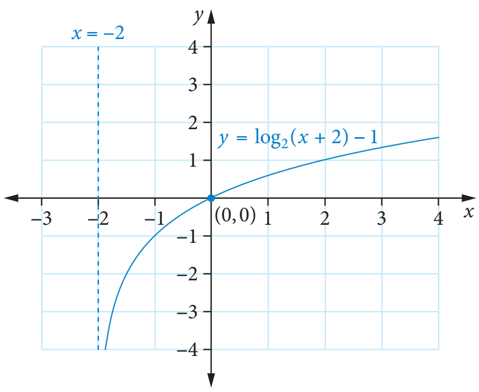


5 Vertical translation 1 unit down and horizontal translation 2 units left.



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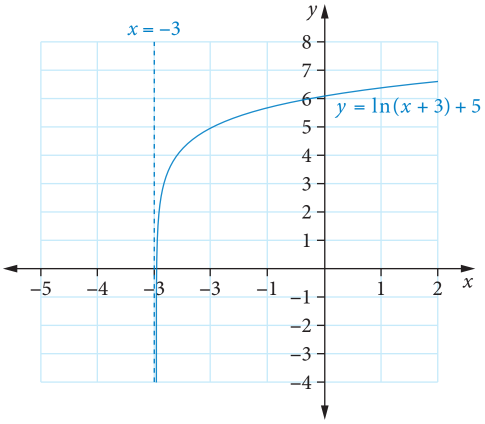
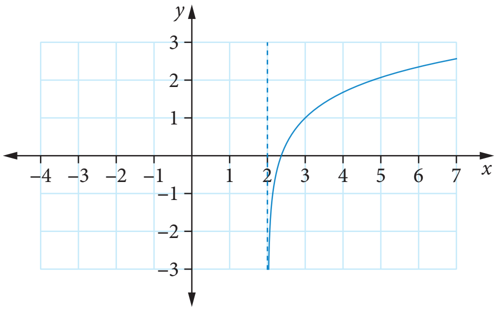


7 $y = 3\ln(x+4)$

8 $b = 7, c = 1$

9

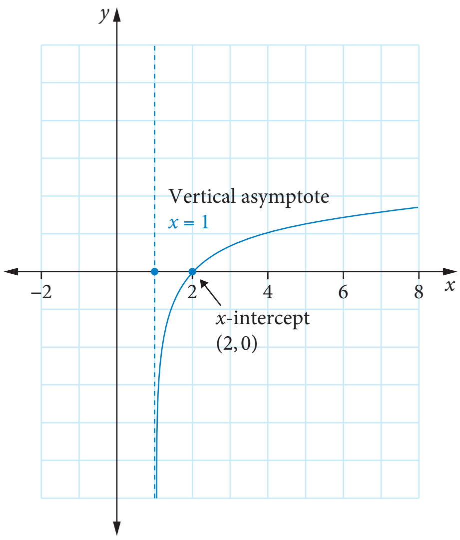


10 **a** x-intercept $(72, 0)$, y-intercept $(0, -2)$, vertical asymptote $x = -9$

b x-intercept $(0, 0)$, y-intercept $(0, 0)$, vertical asymptote $x = -8$

11 $y = \log_5(x-2)+8$

12 **a**



b $m = a + 1$ **c** $(a^{-c} + 1 - b, 0)$

13 $m = -4, p = 5, q = 2$

EXERCISE 6.4

1 $b = 20, c = 10$ **2** $b = 2, c = 8$

3 **a** $n = 3.7$ **b** $n = 2.5$

4 **a** \$507 **b** 58 sheep

c between 14 and 138 sheep

5 **a** 549 **b** 3225

c $t = 2.457$, that is 4 January

6 The amplitude of the waves in Mexico is $10^{1.1}$(or 12.59) times higher than that of the waves in San Francisco Bay.

7 **a** **i** 100 **ii** 1000

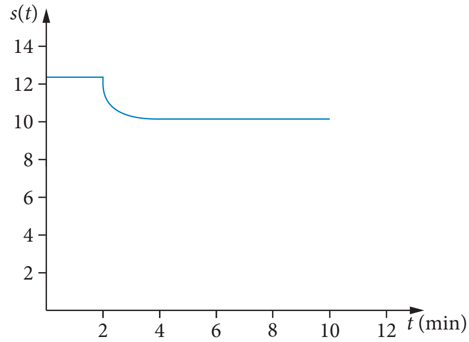
b $A = 0.5, B = 2$

c 100 000

d $N = 100(10)^{0.5t}$

8 **a** $p = 4$ **b** $p = 0.1$

9 **a**



b She runs at 10 km/h when she has run for 2.99 minutes.

c She has zero acceleration for the first 2 minutes of her run and at the instant $t = 6.30$ minutes.

10 $\dfrac{A_{NZ}}{A_H} = \dfrac{10^{5.5}}{10^{3.4}} = 10^{2.1}$

11 **a** 97 dB

b $I_0 = \dfrac{1 \times 10^{-5}}{10^7} = 1 \times 10^{-12}$ watt/m^2

CUMULATIVE EXAMINATION: CALCULATOR-FREE

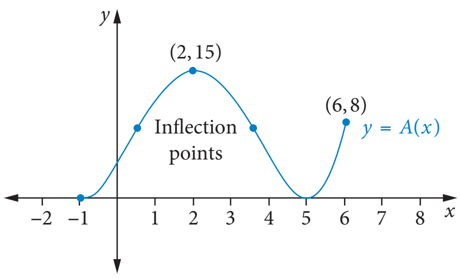
1 $\dfrac{dy}{dx} = 1 + x(x^2-4)^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = (x^2-4)^{-\frac{1}{2}} - x^2(x^2-4)^{-\frac{3}{2}}$

$\text{LHS} = (x^2-4)\left((x^2-4)^{-\frac{1}{2}} - x^2(x^2-4)^{-\frac{3}{2}}\right)$

$+ x\left(1 + x(x^2-4)^{-\frac{1}{2}}\right) - \left(x + (x^2-4)^{\frac{1}{2}}\right) = 0 = \text{RHS}$

2



3 **a** $\dfrac{\ln(11)-3}{2}$ **b** $\ln(3)$

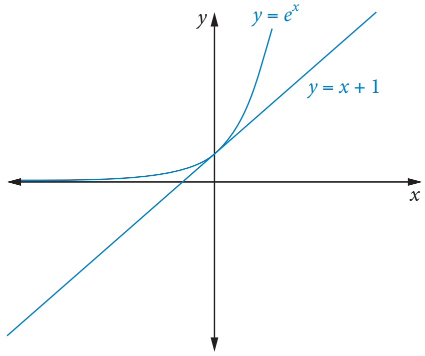
c $\dfrac{18}{3}$ **d** $\dfrac{e^3-2}{4}$

4 $a = -6, b = 4$

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 **a** $(0, 1)$ **b** $c = 1$

c

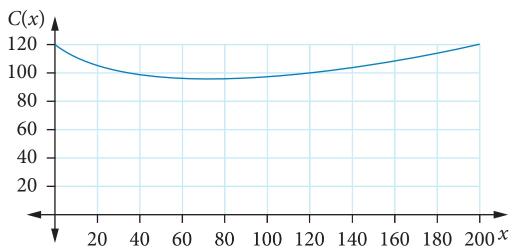


d $1 - \dfrac{(\ln(2))^2}{2} - \ln(2)$ **e** area $= -\int_{\ln 2}^{1} \ln(x)\,dx$

2 **a** 0.486 **b** 0.017 **c** 0.312

3 **a** $\$10\,475.71 \approx \$10\,476$

b



c The minimum is at $x = 74.205$.

$C(74) = 95.4307$ i.e. \$9543.07

$C(75) = 95.4320$ i.e. \$9543.20

The company should manufacture 74 components.

4 **a** Since $P(0) = 4$, we require $\ln(a) = 1$, giving $a = e$.

b Profit will be approximately \$4 087 000.

c Maximum profit is approximately \$4 436 000. This occurs when $x \approx 1.79$, so during the second week.

d The model predicts during the 6th week.

e Profit should exceed \$5 million during the 8th week after the changes.

CHAPTER 7

EXERCISE 7.1

1 **a** $\dfrac{8}{8x - 5}$

b $\dfrac{6x + 6}{3x^2 + 6x} = \dfrac{2x + 2}{x^2 + 2x}$

c $\dfrac{12x^3 + 24}{x^4 + 8x}$

2 **a** $\dfrac{1}{x}$ **b** $\dfrac{3}{x}$ **c** $\dfrac{8}{4x - 3}$

d $\dfrac{1}{4(x - 4)}$ **e** $\dfrac{6}{2x + 1}$

3 **a** $(2x - 2)\ln(x) + x - 2$

b $3x^2 + 9x^2 \ln(x)$

c $\dfrac{-\ln(x) + 1}{x^2}$

4 $\dfrac{x^2 - 3x^2\ln(2x)}{x^6} = \dfrac{1 - 3\ln(2x)}{x^4}$

5 **a** $\dfrac{-25}{(5x + 4)^2}$ **b** $\dfrac{-64}{(4x + 1)^2}$

6 **a** $\dfrac{4}{4x - 3}$ **b** $\dfrac{-16}{(4x - 3)^2}$

7 $\dfrac{2\cos(2)}{e}$

8 **a** Proof: see worked solutions **b** $-\dfrac{5}{24}$

9 $x + 2x\ln(x)$

10 $1 + \ln(x)$

11 **a** $\dfrac{1 - 2\log_e(x)}{x^3}$ **b** 1

12 0.8

13 $x^2 + 3x^2\ln(2x)$

EXERCISE 7.2

1 **a** $\dfrac{dy}{dx} = \dfrac{-2}{5 - 2x}$

b $\dfrac{dy}{dx} = \dfrac{3x^2 + 2x}{x^3 + x^2} = \dfrac{3x + 2}{x^2 + x}$

2 $\dfrac{-1}{(x + 6)^2}$

3 **a** $(4, \ln 16)$

b $f''(x) = \dfrac{-2x^2 + 16x - 64}{(8x - x^2)^2}$, $f''(4) = -\dfrac{1}{8}$, local maximum

4 **a** $N'(t) = \dfrac{400(6 - t)}{(-t^2 + 12t + 13)}$

b local maximum at $t = 6$ weeks

c 778 frogs

5 $y = \dfrac{x}{e^2} + 2$

6 $y = \dfrac{1}{3}x - 1 + \ln(3)$ $\dfrac{-72}{5(2t + 1)^2}$

7 $y = 3x - 9$

8 $\ln(3.003) \approx 1.0996$

9 **a** $v(t) = \dfrac{16}{2t + 1}$ **b** 1.5 hours **c** $-\dfrac{32}{9}$ km/h^2

10 **a** The company will make a loss for a selling price between \$1.50 and \$2.00. The profit then increases to approximately \$2.25 per item sold for a selling price of approximately \$3.25, and then decreases steadily to a value of less than \$1 per item sold for a selling price of \$10.

b $x = 2e^{\frac{1}{2}}$

11 $1 + \ln(2x)$

12 $\ln(2) + 0.01 = 0.703$

13 a $x(1+2\ln x)$

b $\dfrac{dy}{dx} = x(1+2\ln x)$

$\dfrac{dy}{dx} = 0,\ \ln x = -\dfrac{1}{2},\ x \neq 0$

Only one point where derivative is zero hence only one stationary point.

14 a $\dfrac{36}{3(2t+1)}$ **b** $-\dfrac{72}{5(2t+1)^2}$

c $-\dfrac{72}{125}$ km/h^2 **d** 1 h 24 min

EXERCISE 7.3

1 $e^x\ln(x) + \dfrac{e^x}{x}$ **2** $\dfrac{4}{3}$

3 a $2\ln(x)+c$ **b** $\dfrac{6}{5}\ln(x)+c$ **c** $\dfrac{1}{3}\ln(x)+c$

4 a $\ln(x^2+11x-15)+c$

b $5\ln(x^3-13)+c$

c $2\ln(3x^3+4x^2+1)+c$

5 a $\ln(x-6)-\ln(x-5)+c$

b $\dfrac{1}{4}\ln(2x-5)-\dfrac{1}{4}\ln(2x+5)+c$

6 a $2x\ln(2x)+x$ **b** $\dfrac{1}{2}x^2\ln(2x)-\dfrac{1}{4}x^2+c$

7 a $3+\log_e(x^3)$ **b** $x\log_e(x^3)-3x+c$

8 a $\dfrac{1}{5}\ln(5x+3)+c$ **b** $\dfrac{3}{2}\ln(2x-5)+c$

9 a $\ln(5)$ **b** $3\ln(2)$ **c** $\dfrac{1}{3}\ln\left(\dfrac{19}{16}\right)$

d $\dfrac{1}{3}\ln\left(\dfrac{7}{4}\right)$ **e** $2\ln(2)$

10 $y = \dfrac{7}{3}\ln(3x-5)+7$

11 $y = 9\ln(x-3)-4x-11$

12 $m = \dfrac{5e^7+1}{3}$ **13** $k = \dfrac{9}{2}$

14 a $\log_e(3x)+1$ **b** $\log_e(12)$

15 a $b = \dfrac{9}{2}$ **b** $p = \dfrac{1}{2}$

16 a $1+\ln(x)$

b $\dfrac{d}{dx}(x\ln(x)) = 1+\ln(x)$

$\displaystyle\int \frac{d}{dx}(x\ln(x))\,dx = \int(1+\ln(x))\,dx$

$\displaystyle x\ln(x) = x + \int \ln(x)\,dx + c$

$\displaystyle\int \ln(x)\,dx = x\ln(x) - x + c$

17 a Proof: see worked solutions

b $f(x) = 6\ln(x-1)+x-1$

c 7 **d** $y = 7x-13$

EXERCISE 7.4

1 $m = 3$ **2** $f(x) = \ln(x+3)+12$

3 $\dfrac{1}{3}\ln(2)$ units^2 **4** 12 units^2

5 a $b = \ln(7)$ **b** $7\ln(7)-6$ units^2

6 a $a = 2$ **b** e^2-1

7 a $2x\log_e(x)+x$ **b** $\dfrac{9}{2}\ln(3)-2$ units^2

8 a i $(\ln 2, 2)$

ii $2\ln(2)-1$ units^2

b $k = \ln 3$

9 a 1 unit^2 **b** $2\ln(2)-1$ units^2

c $a\ln(a)-a+1$ units^2

10 a 549 **b** 3225

c $t = 2.457$, that is 3 January

11 a $a = 1.949$ **b** 168 days

c 0.012 s/day

d It would take him 1314 days, or more than $3\frac{1}{2}$ years. This is probably impossible to maintain.

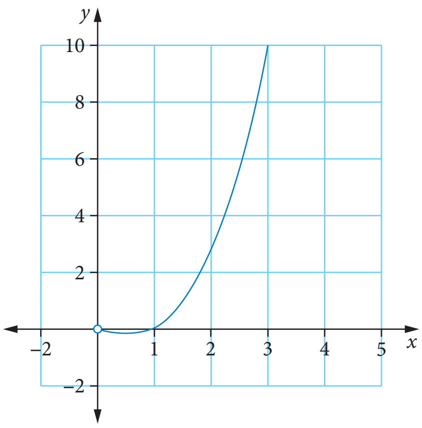
12 a $a = 1.820$

b 15 weeks

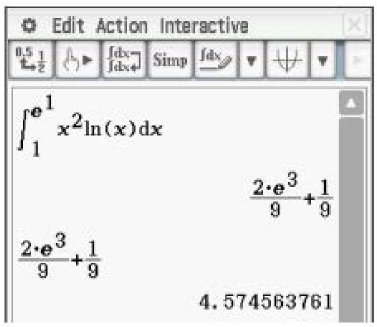
c At $t = 4$, $\dfrac{dN}{dt} = 0.063$ skateboards per day.

d At $t = 10$, $\dfrac{dN}{dt} = 0.165$ skateboards per day.

13 a



b



14 a i $-\dfrac{7}{a}$ **ii** $x = \sqrt{a}$

b i 7 **ii** $b = e^{-1}$

c i $\dfrac{7(a^2-1)}{2a}$ **ii** $a = \sqrt{2}+1$

iii The area under the curve is less than the area of the trapezium.

Hence $\int_1^a f(x)\,dx < 7$.

From part **b i**

$\int_1^e f(x)\,dx = 7$ but $\int_1^a f(x)\,dx < 7$, so $a < e$.

d $m = e^{\frac{5}{14}}, n = e^{\frac{1}{14}}$

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 **a** $d_0 e^0 = 2$ and $d_0 e^{2m} = 10$ **b** $d_0 = 2$, $m = \frac{1}{2}\ln(5)$

2 **a** $p = \frac{1}{3}$

b

x	0	1	2	3
$P(X = x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

c $\frac{26}{27}$ **d** $\frac{6}{13}$

3 $\left(\ln\left(\frac{3}{2}\right), 0\right)$ and $(\ln(2), 0)$ **4** $\frac{1}{2}\ln(6)$

5 **a** **i** Two distinct cases in which the upper bound is twice the lower bound. Shade under the curve from $x = 1$ to $x = 2$, and then from $x = 2$ to $x = 4$.

Other possibilities would be $x = 1.5$ to $x = 3$, $x = 2.5$ to $x = 5$ or $x = 3$ to $x = 6$.

ii $b = 3a$

b **i** Using the rectangles that estimate $y = \frac{1}{x}$ on the left side of each interval gives

$\int_2^3 \frac{1}{x}\,dx < \frac{1}{2} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{2} = \frac{9}{20}$.

This is an overestimate of the integral as the top of the rectangles lie above the graph.

Using the rectangles that estimate $y = \frac{1}{x}$ on the right side of each interval gives

$\int_2^3 \frac{1}{x}\,dx > \frac{2}{5} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{11}{30}$.

This is an underestimate of the integral as the top of the rectangles lie below the graph.

Hence,

$\frac{11}{30} < \int_2^3 \frac{1}{x}\,dx < \frac{9}{20}$.

ii $\int_2^3 \frac{1}{x}\,dx = [\ln(x)]_2^3 = \ln(3) - \ln(2) = \ln(1.5)$

Hence

$\frac{11}{30} < \ln(1.5) < \frac{9}{20}$.

6 $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x - 2) - 1 + \frac{1}{2}\log_e(2)$

7 **a** $2xe^{kx} + kx^2e^{kx} = xe^{kx}(kx + 2)$

b $k = 1$

c $\int_0^2 x^2e^{kx} + \frac{2xe^{kx}}{k}\,dx$

d $k = \ln(2)$

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 **a** $x = \frac{1}{a}$ and $x = \frac{ab+1}{4a}$ **b** $a = 0$

c $a = \frac{3}{b}$ **d** 2 **e** $p = 4$

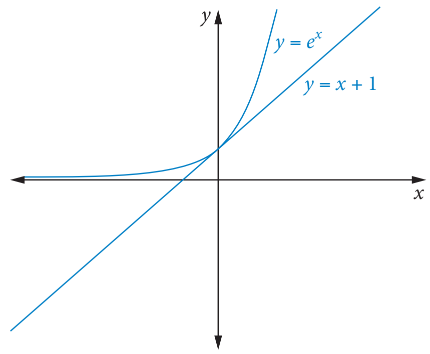
2 -1

3 **a** $f'(x) = x^3 + 4x^3\ln(4x)$ **b** $\frac{1}{4}x^4\ln(4x) - \frac{1}{16}x^4 + c$

c $\frac{1}{2}\ln(2) - \frac{225}{4096}$ **d** 28 m

4 **a** $(0, 1)$ **b** $c = 1$

c



d area $= 1 - \frac{(\ln(2))^2}{2} - \ln(2)$

e area $= -\int_{\ln 2}^{1} \ln(x)\,dx$

CHAPTER 8

EXERCISE 8.1

1 **a** $n(20 \le t < 25) = 3$, $n(35 \le t < 40) = 6$

b **i** $\frac{2}{28}$ **ii** $\frac{2}{8}$ **iii** $\frac{20}{28}$ **iv** $\frac{20}{28}$

v $\frac{3}{5}(3) + \frac{3}{5}(2) = \frac{15}{5} = 3$, so $\frac{3}{28}$.

2 $k = \frac{6}{125}$

3 **a** 0 **b** $\frac{4}{9}$ **c** $\frac{1}{3}$ **d** $\frac{3}{4}$

4 **a** $\frac{15}{16}$ **b** $\frac{1}{\sqrt{e}} - \frac{1}{\sqrt[5]{e^4}}$ **c** $\frac{1}{3}$

5 $a = 5$ (reject -5 as $a > 0$)

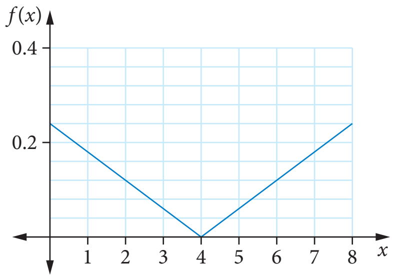
6 **a** 0.68 **b** 0.36 **c** 0.64

7 **a** $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{9} & 0 \le x \le 3 \\ 1 & x > 3 \end{cases}$ **b** $F\left(\frac{5}{2}\right) = \frac{25}{36}$

8 a $\frac{3}{8}$

b $f(x) = \begin{cases} \frac{1}{4} - \frac{x}{16} & 0 \le x < 4 \\ \frac{x}{16} - \frac{1}{4} & 4 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$

c



d $\int_0^4 \frac{1}{4} - \frac{x}{16} dx + \int_4^x \frac{t}{16} - \frac{1}{4} dt$

$= \frac{1}{2} + \left[\frac{t^2}{32} - \frac{t}{4}\right]_4^x$

$= \frac{1}{2} + \frac{x^2}{32} - \frac{x}{4} - \frac{1}{2} + 1$

$= \frac{x^2}{32} - \frac{x}{4} + 1$

9 a 0.1 **b** 0.26 **c** 0.008

10 a 0.12 **b** $\frac{1}{28}$ **c** 10% **d** $k = \frac{27}{560}$

11 a $\int_0^4 \frac{x+1}{k} dx = \frac{1}{5}\int_0^4 x + 1\, dx$

$= \frac{1}{k}\left[\frac{x^2}{2} + x\right]_0^4$

$= \frac{1}{k}[8 + 4]$

$\frac{12}{k} = 1$

$k = 12$

b $k = 3$ (reject -5 as $k > 0$) **c** $\frac{7}{15}$

12 a $\lim_{k\to\infty}\left(\int_a^k \frac{a}{x^2} dx\right) = \lim_{k\to\infty}\left(\left[-\frac{a}{x}\right]_a^k\right) = \lim_{k\to\infty}\left(-\frac{a}{k} + 1\right)$

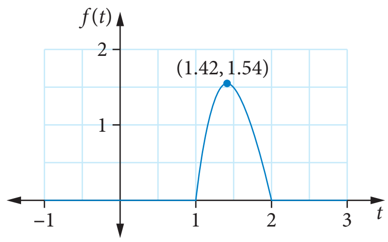
As $k \to \infty$, $-\frac{a}{k} + 1 \to 1$. All values of $f(x)$ are positive and sum of all probabilities equals 1. Therefore, a valid pdf.

b $1 - P(X \le 2a) = 1 - \int_a^{2a} \frac{a}{x^2} dx = \frac{1}{2}$

13 $\frac{7}{12}$ or 0.5833 (4 d.p.)

14 a $a = -\frac{3}{4}$ **b** 0.5248 (4 d.p.)

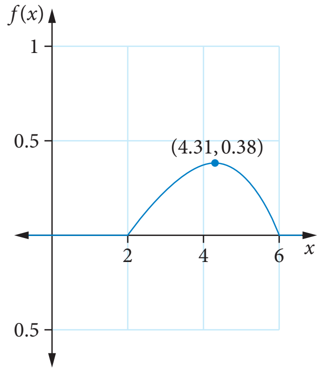
15 a



b 0.191

c $k = 1.77$ h, $k \approx 106$ minutes

16 a



b 0.1211

17 a 14.66

b $\int_k^{k+1} \cos(t) + 1dt = [\sin(t) + t]_k^{k+1}$

$= \sin(k+1) + k + 1 - \sin(k) - k$

$= \sin(k+1) - \sin(k) + 1$

$1 = \sin(k+1) - \sin(k) + 1$

$\sin(k+1) = \sin(k), 0 < k < 2$

$k = 1.070796327\ldots$

$= \frac{\pi - 1}{2}$

18 a $\int_0^4 \frac{y}{16} dy + \lim_{k\to\infty}\left(\int_4^k 0.25e^{-0.5(y-4)} dy\right)$

$= \left[\frac{y^2}{32}\right]_0^4 + \lim_{k\to\infty}\left(\left[-0.5e^{-0.5(y-4)}\right]_4^k\right)$

$= 0.5 + \lim_{k\to\infty}\left(-0.5e^{-0.5(k-4)} + 0.5e^0\right)$

$= 0.5 + \lim_{k\to\infty}\left(-0.5e^{-0.5(k-4)} + 0.5e^0\right)$

$= 0.5 + 0 + 0.5$

$= 1$

All values of $f(y)$ are positive and the sum of all probabilities equals 1. Therefore, it is a valid pdf.

b 0.4155

EXERCISE 8.2

1 D **2** E **3** 6 **4** $\frac{14}{9}$

5 a $\int_0^1 k\sin(\pi x)\, dx = \left[-\frac{k}{\pi}\cos(\pi x)\right]_0^1$

$= \left(-\frac{k}{\pi}\cos(\pi) + \frac{k}{\pi}\cos(0)\right)$

$= \frac{k}{\pi} + \frac{k}{\pi}$

$= \frac{2k}{\pi}$

$\frac{2k}{\pi} = 1$

$k = \frac{\pi}{2}$

b $\frac{1}{2}$

6 $m = \sqrt{13}$ (reject $-\sqrt{13}$ as $m > 1$) or 3.61 (2 d.p.)

9780170477536

7 $k = \frac{2\sqrt{30}}{5}$ (reject $-\frac{2\sqrt{30}}{5}$ as $k > 0$) or 2.19 (2 d.p.)

8 $\text{Var}(X) = \frac{1}{18} = 0.0\dot{5}$

$\text{SD}(X) = \frac{\sqrt{2}}{6} = 0.2357$

9 **a** $\text{E}(Y) = \frac{2}{3}(3) - 1 = 1$

b $\text{Var}(Y) = \frac{4}{9}(9) = 4$

c $\text{SD}(Y) = 2$

10 **a** $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$

$$\begin{aligned}\text{E}(X^2) &= \text{Var}(X) + \text{E}(X)^2\\ &= 5 + 2^2\\ &= 9\end{aligned}$$

b 8

c $\frac{1}{5}$

11 **a**
$$\begin{aligned}\int_1^2 \frac{k}{x^2}\,dx &= \left[-\frac{k}{x}\right]_1^2\\ &= -\frac{k}{2} + \frac{k}{1}\\ &= -\frac{k}{2} + \frac{2k}{2}\\ &= \frac{k}{2}\\ \frac{k}{2} &= 1\\ k &= 2\end{aligned}$$

b $2\ln(2)$

12 $5\ln(2)$

13 **a**
$$\begin{aligned}\int_0^5 ax(5-x)\,dx &= a\int_0^5 5x - x^2\,dx\\ &= a\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5\\ &= a\left(\frac{125}{2} - \frac{125}{3}\right)\\ &= a\left(\frac{125}{6}\right)\\ \frac{125a}{6} &= 1\\ a &= \frac{6}{125}\end{aligned}$$

b Given that the pdf is a parabola with roots at $x = 0$ and $x = 5$, the line of symmetry has the equation $x = \frac{5}{2}$. So, $f(x)$ is symmetrical about $x = \frac{5}{2}$ over $0 \le x \le 5$ and, hence, $\text{E}(X) = \frac{5}{2}$.

c $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$
$$\begin{aligned}&= \frac{6}{125}\int_0^5 5x^3 - x^4\,dx - \left(\frac{5}{2}\right)^2\\ &= \frac{6}{125}\left[\frac{5x^4}{4} - \frac{x^5}{5}\right]_0^5 - \frac{25}{4}\\ &= \frac{6}{125}\left(\frac{5^5}{4} - \frac{5^5}{5}\right) - \frac{25}{4}\\ &= \frac{6}{125}\left(\frac{5^5}{20}\right) - \frac{25}{4}\\ &= \frac{6(25)}{20} - \frac{25}{4}\\ &= \frac{6(25)}{20} - \frac{5(25)}{20}\\ &= \frac{25}{20}\\ &= \frac{5}{4}\end{aligned}$$

$\text{SD}(X) = \frac{\sqrt{5}}{2}$

14 $m = 1.2285$

15 **a** $\text{E}(X) = \frac{4}{9} \approx 27$ minutes

b $\text{SD}(X) = \sqrt{\frac{13}{162}} = 0.2833 \approx 17$ minutes

16 **a**
$$\begin{aligned}\lim_{k\to\infty}\left(\int_0^k \frac{1}{8}e^{-\frac{x}{8}}\,dx\right) &= \lim_{k\to\infty}\left(\left[-e^{-\frac{x}{8}}\right]_0^k\right)\\ &= \lim_{k\to\infty}\left(-e^{-\frac{k}{8}} + 1\right)\\ &= 1\end{aligned}$$

All values of $f(x)$ are positive and the sum of all probabilities equals 1. Therefore, it is a valid pdf.

b **i** $\text{E}(X) = 8$ **ii** $\text{Var}(X) = 64$

c $k = 2.301$

17 **a** 1

b $k = 2.5101$ (reject $k = 0.0501$ as $k > 1$)

c $\text{E}(X) = 2.0973$

d $\text{Var}(X) = 0.1760$

e $\text{SD}(Y) = 2\text{SD}(X) = 0.8392$

18 **a** $\text{E}(X) = 3.0458$

b $\text{P}(X > 4) = 0.2617$

$0.2617 \times 200 \approx 52$

19 **a** $\text{E}(W) = 306$ g

b $\text{P}(W < 306) = 0.5248$

c $\text{SD}(X) = 8$ g

d $\text{P}(W > 314) = 0.1792$

EXERCISE 8.3

1 E 2 A

3 **a** 0 **b** $\frac{5}{6}$ **c** $\frac{2}{5}$

4 **a** $b - 7 = 13$

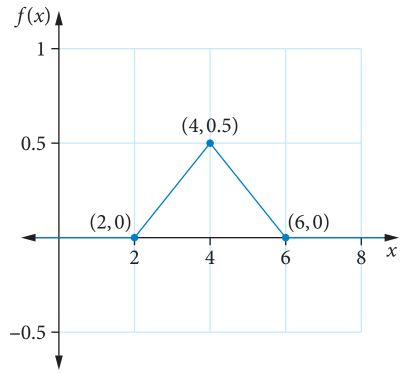
$b = 20$

b $f(x) = \begin{cases} \frac{1}{13} & 7 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$ **c** $\frac{1}{2}$

5 **a** $f(x) = \begin{cases} \frac{1}{50} & -2 \le x \le 48 \\ 0 & \text{otherwise} \end{cases}$ **b** 23

c $\frac{625}{3} = 208.\dot{3}$ **d** $\frac{25\sqrt{3}}{3} = 14.43$

6 **a**



It is a symmetrical triangular distribution about $x = 4$.

b **i** $\frac{1}{8}$ **ii** $\frac{1}{4}$ **iii** 4 **iv** $\frac{2}{3}$

7 **a** $\int_0^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{2} -\frac{2}{3}x + \frac{4}{3}\,dx$

$= \left[x^2\right]_0^{\frac{1}{2}} + \left[-\frac{x^2}{3} + \frac{4x}{3}\right]_{\frac{1}{2}}^{2}$

$= \frac{1}{4} + \left(-\frac{4}{3} + \frac{8}{3} + \frac{1}{12} - \frac{2}{3}\right)$

$= \frac{3 + 16 + 1 - 8}{12}$

$= 1$

All values of $f(x)$ are positive and the sum of all probabilities equals 1. Therefore, it is a valid pdf.

b $\int_0^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{k} -\frac{2}{3}x + \frac{4}{3}\,dx = 0.7$

$\int_{\frac{1}{2}}^{k} -\frac{2}{3}x + \frac{4}{3}\,dx = 0.45$

$k = 1.05$ (reject $k = 2.95$ as $k < 2$)

c $\int_0^{\frac{1}{2}} 2x^2\,dx + \int_{\frac{1}{2}}^{2} -\frac{2}{3}x^2 + \frac{4}{3}x\,dx$

$= \left[\frac{2x^3}{3}\right]_0^{\frac{1}{2}} + \left[-\frac{2x^3}{9} + \frac{2x^2}{3}\right]_{\frac{1}{2}}^{2}$

$= \frac{1}{12} + \left(-\frac{16}{9} + \frac{8}{3} + \frac{1}{36} - \frac{1}{6}\right)$

$= \frac{3 - 64 + 96 + 1 - 6}{36}$

$= \frac{30}{36}$

$= \frac{5}{6}$

8 **a** $\mathrm{E}(L) = 248$ mm

$\mathrm{SD}(L) = 1.73$ mm

b $\mathrm{P}(L > 250) = \frac{1}{6}$

c Let X be the number of pipes with length greater than 250 mm.

$X \sim \mathrm{Bin}\left(60, \frac{1}{6}\right)$

$\mathrm{E}(X) = 10$

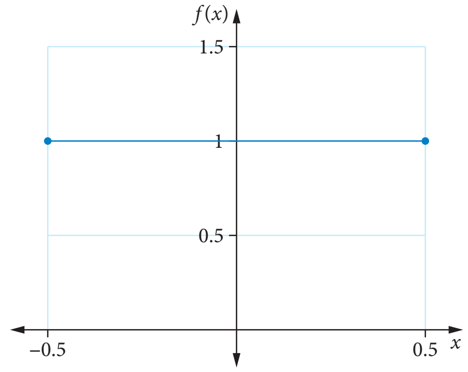
d $\mathrm{P}(X \ge 10) = 0.5536$

9 **a** $f(t) = \begin{cases} \frac{t}{3} & 0 \le t < 1.5 \\ -\frac{t}{5} + \frac{4}{5} & 1.5 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$

b $\mathrm{P}(T < 1) = \frac{1}{6}$

c $\mathrm{P}(1 < T < 3) = \frac{11}{15}$

10 **a**



It is a continuous uniform distribution.

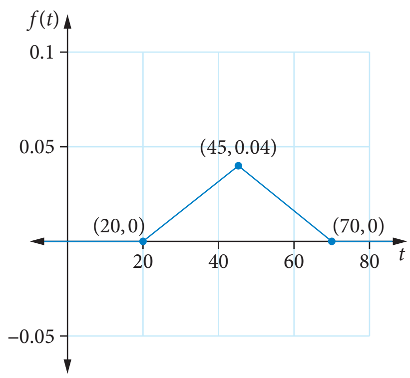
b $\mathrm{P}(X \ge 0.35) = 0.15$

c $\mathrm{P}(X < -0.35 \mid X < 0) = 0.3$

d $\mathrm{P}(X^2 < 0.09) = \mathrm{P}(-0.3 < X < 0.3) = 0.6$

e $\mathrm{Var}(X) = \frac{1}{12}$

11 **a**



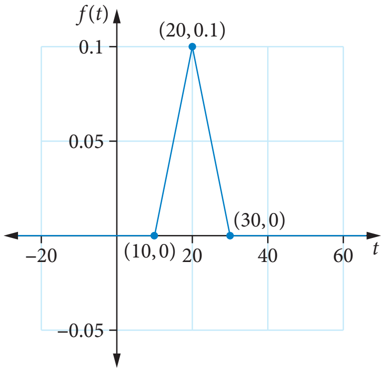
b It is a symmetrical triangular distribution about $t = 45$.

c **i** 0.8

ii 0.0244

iii $k = 39.3649$ (reject $k = 0.6351$ as $k > 20$)

12 **a**



b It is a symmetrical triangular distribution about $t = 20$.

c **i** $\frac{7}{8}$ **ii** $\frac{1}{7}$

13 **a** $\frac{3}{8}$ **b** $\mu = 251$, $\sigma = 2.3094$

c Let X be the number of bottles containing less than 250 mL.

$X \sim \text{Bin}\left(15, \frac{3}{8}\right)$

$P(X = 5) = 0.2025$

14 **a** $E(W) = 249$ g, $SD(W) = 2.3094$

b $P(W > 250) = \frac{3}{8}$

c Let X be the number of packets that weigh more than 250 g.

$X \sim \text{Bin}\left(100, \frac{3}{8}\right)$

$E(X) = 37.5$

approx. 38 packets

d $P(X \geq 25) = 0.9971$

15 **a** Using area under the curve, based on average height

$-a \leq x \leq b$: $(a + b)\left(\frac{3}{4}\right) = 1 \Rightarrow a + b = \frac{4}{3}$.

Using area under the curve, calculating separately for the two parts of the function:

$\frac{1}{2}(a)(2a) + \frac{1}{2}(2a + b)(b) = 1 \Rightarrow a^2 + ab + \frac{b^2}{2} = 1$

b $a = \frac{\sqrt{2}}{3}$, $b = \frac{4 - \sqrt{2}}{3}$

c $1 - P(X \leq 0) = 1 - \left(\frac{\sqrt{2}}{3}\right)^2 = \frac{7}{9}$

EXERCISE 8.4

1 B **2** B

3 **a** 0.95 **b** 0.16 **c** 0.0015

4 **a** Three standard deviations give 2100 kg and $1500 - 2100 = -600$ gives a negative mass.
Not appropriate, as a mass of 0 kg is 2.14 standard deviations below the mean.

b The number of followers is a discrete variable.
Not appropriate, as not continuous.

5 **a** 0.8186 **b** 0.64

6 **a** **i** $z = -3$ **ii** $z = 2$ **iii** $z = 3.5$

b $1 - 0.975 = 0.025$

7 **a** 0.0056 **b** 0.2525 **c** 0.6107

8 **a** $c = 2166.86$ **b** $c = 2245.87$

9 $z = -0.5999$, $\mu = 170 - 50(-0.5999) = 199.99 \approx 200$

10 **a** 0.9332 **b** 0.3694 **c** 0.3959

d Let X be the number of eggs weighing more than 69 g being classified as Jumbo.

$X \sim \text{Bin}(80, 0.3959)$

$P(X = 40) = 0.0151$

11 **a** 0.5 **b** $b = -\frac{2}{3}$

12 **a** $b = -2$ **b** $\frac{0.34}{0.5} = 0.68$

13 **a** 0.16 **b** 0.34 **c** $\frac{0.16}{0.5} = 0.32$

14 $q - \frac{1}{2}$

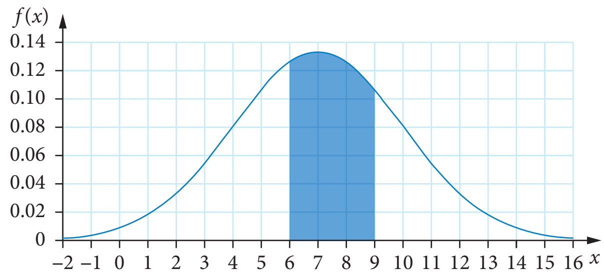
15 **a** Her statement is valid as 142 cm is three standard deviations below the mean and 184 cm is three standard deviations above the mean. Therefore, approximately 99.7% of women will have heights in that range, i.e. almost all.

b 0.16%

c 149 cm

16 **a** **i** 2 **ii** C, as it has the largest spread.

b **i**



ii No. The total area below the probability density function is 1, and the region shaded above is less than half of that area (i.e. area is less than 0.5). Hence, it corresponds to a probability that is less than 0.5.

c Not normal: a continuous random variable has $P(Y \geq 2) = P(Y > 2)$. Since a normally distributed random variable is continuous, it follows that Y is not a normally distributed random variable. It could be binomial, as $P(Y \geq 2) = P(Y > 2)$ for a discrete random variable, as long as $P(Y = 2) \neq 0$. Since the binomial distribution is discrete, it follows that Y could be a binomially distributed random variable.

17 **a** 5.3 g **b** 0.6530

18 **a** $\mu = 81.71$ minutes, $\sigma = 16.24$ minutes

b 0.3554

19 **a** 0.0808 (or 8.08%)

b 0.9832 (or 98.32%)

c 76.45 g

20 a 0.0062

b Let X be the number of pizzas that are delivered free.

$X \sim \text{Bin}(50, 0.0062)$

$\text{P}(X > 3) = 0.0003$

c 31.2 minutes **d** 1.6 minutes

21 a 34.51%

b

Classification	Weight W (grams)	P(W)
Small	$W \le 110$	0.1418
Medium	$110 < W \le 155$	0.5131
Large	$155 < W \le 210$	0.3310
Extra large	$W > 210$	$0.3451 - 0.3310$ $= 0.0141$

c 0.2912

d Let Y be the number of small carrots in a bag.

$Y \sim \text{Bin}(12, 0.1418)$

$\text{P}(Y \le 2) = 0.7637$

22 a $a = 1.0134 \approx 1$ minute **b** 0.5467

c $k = -1.5$ or $k = -2.5$ **d** 0.0029

e i $1 - 0.85^n$ **ii** $n = 19$

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 a $\frac{1}{12}$ **b** $\frac{1}{3}$ **c** $\frac{1}{4}$

2 a

w	0	1	2	3
$\text{P}(W = w)$	0.2	0.4	0.3	0.1

b i $\text{E}(W) = 1.3$ **ii** $\text{Var}(W) = 0.81$

3 a $x = 4.5$ **b** $x = 1.6$

4 a $\frac{8}{x} + c$ **b** $3\ln(x^2 + 5x - 11) + c$

c $\ln(5) - \ln(2)$

5 a $x\frac{\pi}{4}\cos\left(\frac{\pi}{4}x\right) + (1)\sin\left(\frac{\pi}{4}x\right)$

$= \frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{4}\right)$

b
$$\begin{aligned}\text{E}(X) &= \int_0^2 x\frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx \\ &= \int_0^2 \frac{d}{dx}\left(x\sin\left(\frac{\pi}{4}x\right)\right) - \sin\left(\frac{\pi x}{4}\right)dx \\ &= \left[x\sin\left(\frac{\pi}{4}x\right) + \frac{4}{\pi}\cos\left(\frac{\pi}{4}x\right)\right]_0^2 \\ &= 2\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi}\cos\left(\frac{\pi}{2}\right) - 0 - \frac{4}{\pi} \\ &= 2 - \frac{4}{\pi}\end{aligned}$$

6 a It is a continuous uniform distribution.

b i $p = 15, q = 40, k = \frac{1}{25}$

ii 27.5 minutes

iii $\frac{2}{5}$

c $\frac{3}{13}$ **d** 7:49 am

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 a
$$\begin{aligned} SA &= 2\left(\frac{5x^2}{2} + hx + \frac{5xh}{2}\right) \\ 6480 &= 5x^2 + 7xh \\ 7xh &= 6480 - 5x^2 \\ h &= \frac{6480 - 5x^2}{7x}\end{aligned}$$

b $0 < x < 36$

c $\frac{dV}{dx} = -\frac{75}{14}x^2 + \frac{16\,200}{7}$

d $x = 12\sqrt{3}$, $h = \frac{120\sqrt{3}}{7}$

2 a 25°C **b** 76.68°C

c 223°C **d** 8.63°C/min

e As time increases, the rate of change in the temperature of the water → 0.

The temperature of the water → the constant value of T_0.

3 a $\frac{9}{11}$ cm/s

b $h'(t)$ is of the form $\frac{f'(x)}{f(x)}$ (the numerator is the derivative of the denominator), so the function $h(t)$ is the natural logarithm of the denominator.

Also, $+\,c$ needs to be included in the function, as any constant could be included here.

c
$$\begin{aligned}\Delta h &= \int_0^2 \frac{4t + 1}{2t^2 + t + 1}dt \\ &= \ln(11)\,\text{cm}\ (2.398)\end{aligned}$$

d 5 seconds

4 a 0.0228

b Let X be the number of days that the process takes more than 2 hours.

$X \sim \text{Bin}(5, 0.0228)$

$\text{P}(X = 2) = 0.004\,83$

c i

Job duration T (minutes)	$T \le 60$	$60 < T < 120$	$T \ge 120$
Probability	0.0228	0.9545	0.0228
Cost Y ($)	200	600	1200

ii $604.55 **iii** $108.67

iv $\mu = 604.55a + b$, $\sigma = 108.67a$

5 a 24.2%

b Let M be the number of parcels that weigh more than 3.7 kg.

$M \sim \text{Bin}(20, 0.241\,96)$

$\text{P}(M \ge 10) = 0.010\,95$

c

x	≤ 1	$1 < x \le 2$	$2 < x \le 3$	$3 < x \le 4$	$x > 4$
y	$5	$6.50	$8	$9.50	$12
$\text{P}(Y = y)$	0.022 75 (accept 0.021 40)	0.135 91	0.341 34	0.341 34	0.158 66

d \$8.87

e 1.7471

f $\mu = 11.65, \sigma = 2.10$

g Three standard deviations below the mean is a weight of 0. $P(X < 0) > 0$, there is a non-zero probability (small) that the weight will be negative.

CHAPTER 9

EXERCISE 9.1

1 **a** all pies produced at the local pie factory

b 6

c mode = 110 g, median = 103.5 g, mean = 104.3 g, range = 12 g

2 Survey every 10th customer who check-out. Other answers are possible.

3 **a** **i** Spatial bias: people are being asked about flying at the airport and only one airport in the country is surveyed.

ii Use a random number generator to ask randomly selected people upon their check-in/bag drop at various airports around the country. Other answers are possible.

b **i** Temporal bias: customers are only asked on one day for a one-hour window of time.

ii Ask every 15th customer in line at the self-service check-out across a larger time frame, or across multiple days, or across multiple stores. Other answers are possible.

4 Answers will vary depending on simulation. $E(X) = 45$, $SD(X) = 17.32$

5 Answers will vary depending on simulation. $E(Y) = 40$, $SD(Y) = 2.5$

6 Answers will vary depending on simulation. $E(Z) = 0.35$, $SD(Z) = 0.48$

7 Answers will vary depending on simulation. $E(T) = 25$, $SD(T) = 4.08$

8 **a** The method is biased due to the people being asked a leading question; the specific time and location used for the survey.

b In this case the question is not biased; however, only mobile phone users were selected, causing bias. Also, many of these people may just hang up, causing non-response bias.

9 Tina could use a random number generator and pick the sample using the numbers she obtains. Other answers are possible.

10 **a** Any two of the following three reasons:

1 Spatial: only one location, so only those present in that mall will be sampled from.

2 Temporal: lunchtime, so only those present at lunchtime will be sampled from.

3 Selection scheme: quota sampling means that the first 400 workers only are in the sample, so this is not a random sample from all workers.

b Either of the following two reasons:

1 Only those with listed telephone numbers will be selected.

2 Non-response bias: Not everyone will answer their phone when called.

c Amir's, as it samples randomly from the population of workers.

11 1 Temporal: the sample taken is at a fixed time, so only people around at that time will be sampled.

2 Spatial: the location is fixed, so only people at that location will be sampled or not everyone from the suburb will pass by that area, so this is not a random sample of the residents.

12 **a** The shapes of the distributions are seemingly different, and the ranges of values that the variable takes are not consistent in the samples (e.g. Sample A has scores between 8 and 18, Sample B has scores between 74 and 83 and Sample C has scores between 20 and 65).

b **i** Sample A

ii $p < 0.5$, as the distribution is positively skewed.

c **i** Sample B

ii approximately 80

d **i** Sample C

ii The distribution of the sample should become more uniform; that is, there could be less variation in the heights of the columns and a 'flatter' distribution.

EXERCISE 9.2

1 **a** supporters of a local football team

b Any of the following reasons:

1 Spatial: only supporters at the game were asked.

2 Spatial: groups of people standing together were asked, meaning they could have travelled together using the same mode of transport.

3 Temporal: data is only collected from one match.

2 C

3 **a** $\frac{28}{156} = 0.1795 = 17.95\%$

b approx. 176 students

4 **a** 0.8 **b** 0.0013 **c** 0.036

5 **a** $\frac{9}{200} = 0.045 = 4.5\%$

b **i** 0.045 **ii** 0.0002 **iii** 0.015

6 18 000

7 Answers will vary depending on simulation

a The distribution should be slightly positively skewed.

b The distribution should be slightly negatively skewed.

c The distribution should be very negatively skewed.

8 Answers will vary depending on simulation

a The distribution should be positively skewed.

b The distribution should be approximately symmetrical.

c The distribution should be negatively skewed.

9 **a** 0.4545

b $E(\hat{p}) = 0.4545$, $SD(\hat{p}) = 0.1062$

c Given that $\hat{p} \approx 0.5$, the distribution is approximately symmetrical and, since there are at least 10 each of positive and negative observations, the distribution is approximately normal with $\hat{p} \sim N(0.4545, 0.1062^2)$ (4 d.p.).

10 **a** Given that n is sufficiently large and that $np = 122 \geq 10$ and $n(1-p) = 378 \geq 10$, $\hat{p}$ is approximately normal with $\hat{p} \sim N(0.244, 0.0192^2)$ (4 d.p.).

b **i** $P(\hat{p} > 0.25) = 0.3774$

ii $P(0.15 < \hat{p} < 0.25) = 0.6226$

11 **a** $\frac{8}{125} = 0.064 = 6.4\%$

b Despite the sufficiently large sample size, $np < 10$, and so the distribution may not be appropriately symmetrical for the normal model. Additionally, $3SD(\hat{p}) = 0.0657$ and so, $0.064 - 3SD(\hat{p}) < 0$ meaning a normal distribution is not appropriate.

c Let X be the number of non-sellable items.

$X \sim \text{Bin}\left(125, \frac{8}{125}\right)$

$P(X > 5) = 0.8178$

d $\hat{p} \sim N(0.064, 0.0219^2)$

$P\left(\hat{p} > \frac{5}{125}\right) = 0.8635$

The normal approximation gives a probability that is 0.05 greater than the discrete binomial calculation.

12 **a** Given that n is sufficiently large and $np = 49.2 \geq 10$ and $n(1-p) = 70.8 \geq 10$, it is appropriate to model $\hat{p}$ using an approximate normal distribution.

b $\hat{p} \sim N(0.41, 0.0449^2)$, $z = 2.00$
i.e. approx. 2 standard deviations above the mean

13 **a** $n = 80$ **b** $k = 1.0811$ **c** $\hat{p} = 0.43$

14 **a** 9

b Given that $np = 9 < 10$, the distribution may not be appropriately symmetrical for an approximate normal distribution.

c $\binom{10}{2}(0.12)^2(0.88)^8$

15 If $\hat{p} = 0$, then $X = 0$.

$\binom{5}{0}p^0(1-p)^5 = \frac{1}{243} \Rightarrow 1 - p = \frac{1}{3} \Rightarrow p = \frac{2}{3}$

16 **a** $n = 25$ **b** $p = \frac{1}{5}$

17 **a** $\frac{35}{60} = 0.58\dot{3} = 58.\dot{3}\%$

b $E(\hat{p}) = 0.58$, $SD(\hat{p}) = 0.0636$

c Given that $p \approx 0.5$ and $n = 60$ is sufficiently large, the distribution of $\hat{p}$ should be approximately normal with a mean of 0.58. The distribution may be slightly negatively skewed.

18 **a** $\hat{p} \sim N(0.02, 0.0082^2)$, $P(\hat{p} \leq 0.03) = 0.888$

b $0.02 - 3SD(\hat{p}) = 0.02 - 3(0.0082) = -0.0047 < 0$ or $np = 5.8 < 10$

c Increase; as $n \to \infty$, $SD(\hat{p}) \to 0$ and so the distribution will become narrower, giving a greater area under the curve for $\hat{p} \leq 0.03$.

19 **a** Given that n is sufficiently large and $p \approx 0.5$, then $\hat{p}$ is approximately normal such that $\hat{p} \sim N(0.6, 0.0219^2)$.

b $P(\hat{p} < 0.58) = 0.1807$

c $\hat{p} = \frac{250}{500} = 0.5$; that is, half of the mangoes.

20 **a** Given that n is sufficiently large and $p \approx 0.5$, then $\hat{p}$ is approximately normal such that $\hat{p} \sim N(0.4, 0.0245^2)$.

b $P(\hat{p} > 0.44) = 0.0512$

21 **a** $\hat{p} = \frac{124}{400} = 0.31 = 31\%$

b Using $p = \frac{7}{24}$, $E(\hat{p}) = \frac{7}{24} = 0.2917$, $SD(\hat{p}) = 0.0227$

c $\hat{p} = 0.2917 \pm 0.6(0.0227)$, $400\hat{p} = 111$ or 122

EXERCISE 9.3

1 D **2** D

3 **a** Given that $\hat{p} = 0.5$, the distribution is approximately symmetrical, and n is sufficiently large ($n \geq 30$), so $\hat{p}$ is approximately normal such that $\hat{p} \sim N(0.5, 0.0707^2)$.

b **i** $0.5 - 1.645\sqrt{\frac{0.5^2}{50}} \leq p \leq 0.5 + 1.645\sqrt{\frac{0.5^2}{50}}$

ii $0.5 - 1.960\sqrt{\frac{0.5^2}{50}} \leq p \leq 0.5 + 1.960\sqrt{\frac{0.5^2}{50}}$

iii $0.5 - 2.576\sqrt{\frac{0.5^2}{50}} \leq p \leq 0.5 + 2.576\sqrt{\frac{0.5^2}{50}}$

4 $[0.614, 0.743]$

5 **a** $[0.312, 0.421]$

b Neither one is more likely than the other to contain p, as the probability that a confidence interval contains p is either 0 or 1. Hence, it cannot be determined.

c approx. 228 confidence intervals

6 **a** If confidence level decreases, z decreases, and so margin of error decreases and, hence, width of the confidence interval decreases.

b If sample size increases, standard error decreases, and so margin of error decreases and, hence, width of the confidence interval decreases.

7 **a** $\hat{p} = 0.483$ **b** $E = 0.075$

c $SD(\hat{p}) = 0.046$ **d** $n \approx 120$

8 $z = 2.572$, $P(-2.572 < z < 2.572) = 0.9899 \approx 99\%$

9 $n \geq 310$

10 a assume worst case scenario and $\hat{p} = 0.5$ for maximum width; $n \geq 1068$

b assume worst case scenario and $\hat{p} = 0.5$ for maximum width; $n \geq 68$

11 a $[0.3669, 0.5931]$. Given that $0.4 \in [0.3669, 0.5931]$, there is insufficient evidence to suggest that the claimed value of p shouldn't be accepted. It could be 40%, but it cannot be known for certain.

b $[0.3285, 0.5049]$. Given that $0.30 \notin [0.3285, 0.5049]$, there is insufficient evidence to suggest that the piano tuning demands north and south of the Swan River are the same.

12 a $[0.2247, 0.5753]$

b $\hat{p} \sim N(0.4, 0.0894^2)$. $P(\hat{p} < 0.24) = 0.0368$. There is a 3.68% chance of randomly selecting a sample in which less than 24% of the people surveyed were in favour of adopting a new state flag.

c A 95% confidence interval using $\hat{p} = 0.24$ and $n = 50$ gives $[0.1216, 0.3584]$. Given that the two confidence intervals overlap with $0.24 \in [0.2247, 0.5753]$, there is insufficient evidence to suggest that the claim is true and that the newly proposed designs weren't better than the current one.

13 $\frac{2}{3} - 1.645\sqrt{\frac{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}{18}} \leq p \leq \frac{2}{3} + 1.645\sqrt{\frac{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}{18}}$

14 4

15 1800

16 a $[0.013, 0.107]$

b The distribution of $\hat{p}$ might not be approximately normal, as $\hat{p} = 0.06$ and $np = 6 < 10$, meaning that the distribution might be positively skewed and, hence, a normal distribution might not be appropriate, or

$0.06 - 3\text{SD}(\hat{p}) = 0.06 - 3(0.0237) < 0$ and so, a normal distribution is not appropriate.

17 $[0.730, 0.789]$

18 As level of confidence increases, z increases and so margin of error increases. Therefore, smallest margin of error will be obtained from the 90% confidence interval. $E = 0.104\,68$

19 a $\hat{p} = \frac{15}{320} = 0.0469 = 4.69\%$

b $[0.0274, 0.0663]$

c 0.0194

20 a $E = \sqrt{\hat{p}(1-\hat{p})}$

$\frac{dE}{dp} = \frac{(1-2\hat{p})}{2\sqrt{\hat{p}(1-\hat{p})}}$

$0 = 1 - 2\hat{p}$

$\hat{p} = \frac{1}{2}$

$\frac{d^2E}{d\hat{p}^2}, \hat{p} = \frac{1}{2} = -2 \Rightarrow$ concave down and so maximum.

b $n \geq 260$

c approx. 63

21 a 9604

b Any two of the following:

Sample size: increasing the sample size, decreasing the width.

Sample proportion: as the sample proportion moves away from 0.5, the width decreases.

Confidence level: as confidence level increases, the width increases.

22 a 9604

b 0.058; that is, within 5.8%

23 a $[0.4562, 0.5438]$

b Given that $0.6 \notin [0.4562, 0.5438]$, then it is unlikely that Tina is correct.

c Take another random sample and obtain another approximate 95% confidence interval for p.

24 a $[0.0227, 0.1773]$

b $[0.0507, 0.1493]$

c Given that 0.06 lies in both confidence intervals, then there is insufficient evidence to suggest that the company's historical data is no longer relevant.

25 a $[0.0562, 0.1038]$

b Given that 0.05 does not lie in the confidence interval, then there is insufficient evidence to suggest that the company's historical data is still relevant; it may be outdated, but it cannot be known for certain.

26 a Given that n is sufficiently large and $np = 43 \geq 19$ and $n(1-p) = 957 \geq 10$, then $\hat{p}$ is approximately normal such that $\hat{p} \sim N(0.043, 0.0064^2)$.

b $[0.030, 0.056]$

c Even though 0.04 lies within the confidence interval $[0.030, 0.056]$, it cannot be inferred that the reviewer visited that specific hall.

27 a $\hat{p} = \frac{56}{250} = 0.224$

b $[0.1723, 0.2757]$

c $E = 0.0517$

d No, even though 0.17 does not lie in the confidence interval $[0.1723, 0.2757]$, it cannot be inferred that the junior staff member made a mistake. Due to the nature of random sampling and the construction of 95% confidence interval, it is expected that 95% of the 95% confidence intervals are expected to contain the true proportion and 5% will not.

28 a The 95% confidence interval will have a larger width. $(0.04, 0.16)$ has a width of 0.12 and $(0.05, 0.15)$ has a width of 0.10. Therefore, $(0.04, 0.16)$ is the 95% confidence interval.

b $\hat{p} = 0.1$ meaning that 10 wins were observed out of 100 games.

c $\sqrt{\frac{p(1-p)}{n}} \rightarrow \sqrt{\frac{p(1-p)}{4n}} \Rightarrow \sqrt{\frac{p(1-p)}{n}} \rightarrow \frac{1}{2}\sqrt{\frac{p(1-p)}{n}}$

Assuming that p remained approximately 0.1, it would be expected that the width of the confidence interval will reduce by a factor of 2 (i.e. halved).

ANSWERS

d No mistake has necessarily been made, as not all confidence intervals are expected to contain the true value of p. 90% of 90% confidence intervals and 95% of 95% confidence intervals are expected to contain the true value of p.

29 a $[0.660, 0.700]$

It is assumed that $\hat{p}$ is approximately normally distributed such that $\hat{p} \sim N(0.68, 0.0123^2)$.

b 0.0201

c Survey 2: $[0.536, 0.565]$, Survey 3: $[0.632, 0.672]$, Survey 4: $[0.670, 0.697]$

d Given that there is no overlap between the confidence intervals for Survey 1 and Survey 2, there is sufficient evidence to suggest that the samples may have come from different populations, i.e. that Survey 2 did not come from Western Australia, but we cannot say for certain.

e $n = 5800$

30 a $X \sim \text{Bin}(30, 0.8)$, $\text{E}(X) = 24$ and $\text{SD}(X) = 2.191$

b $\text{P}(X \geq 27) = 0.1227$

c Given that n is sufficiently large and $np = 450 \geq 10$ and $n(1-p) = 150 \geq 10$, then $\hat{p}$ is approximately normally distributed such that $\hat{p} \sim N(0.75, 0.0177^2)$. Therefore, $[0.7154, 0.7846]$.

d Given that $0.8 \notin [0.7154, 0.7846]$, then there is insufficient evidence to suggest that there isn't an error with the machine's calibration and it may be off, but it cannot be known for certain.

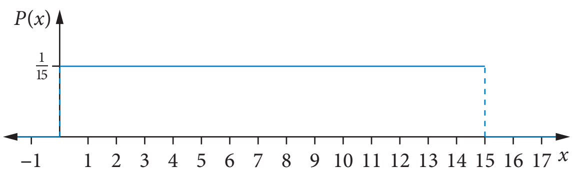
e Not all confidence intervals are expected to contain the true value of p and it is expected that 95% of confidence intervals to be constructed will contain p. It can be due to the nature of random sampling that 0.8 did not lie in the first confidence interval.

CUMULATIVE EXAMINATION: CALCULATOR-FREE

1 a $np = 20$, $np(1-p) = 4$ **b** $n = 25$, $p = \frac{4}{5}$

2 a $3x^2 \ln(3x) + x^2$ **b** $\frac{1}{3}x^3 \ln(3x) - \frac{1}{9}x^3 + c$

3 a



b $\frac{3}{15}$ **c** $\frac{7}{15}$ **d** $\frac{5}{8}$

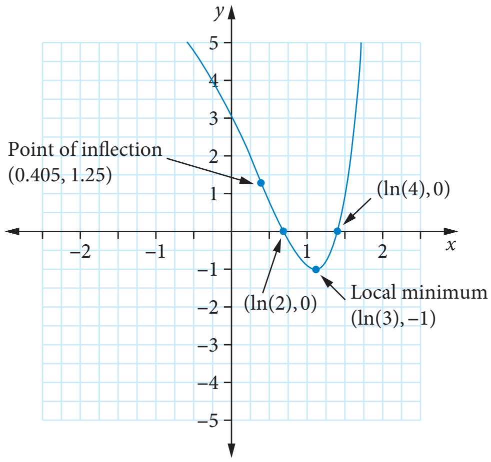
4 a $(\ln(2), 0)$ $(\ln(4), 0)$

b $f'(x) = 2e^{2x} - 6e^x$

$f'(x) = 2e^x(e^x - 3) = 0$, stationary point $(\ln(3), -1)$

c $\left(\ln\left(\frac{3}{2}\right), \frac{5}{4}\right)$

d



CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED

1 0.121 cm

2 a $f'(x) = -0.6(x-4)(x-2)$

b i $y = -1.8x + 2.6$ **ii** $Q\left(\frac{13}{9}, 0\right), S\left(0, \frac{13}{5}\right)$

c 21.6 units^2

3 a i X represents the number of questions Steve answers correctly.

$X \sim \text{Bin}(3, 0.25)$

$\Pr(X = 3) = \frac{1}{64}$

ii $X \sim \text{Bin}(20, 0.25)$

$\text{P}(X \geq 10) = 0.0139$

iii Proof: see worked solutions

b $\frac{17}{64}$

c $X \sim \text{Bin}(25, p)$

$\text{P}(Y > 23) = 6\text{P}(Y = 25)$

$\text{P}(Y = 24) + \text{P}(Y = 25) = 6\text{P}(Y = 25)$

$\text{P}(Y = 24) - 5\text{P}(Y = 25) = 0$

${}^{25}C_{24}\, p^{24}(1-p)^1 - 5 \times {}^{25}C_{25}\, p^{25}(1-p)^0 = 0$

$25p^{24}(1-p) - 5p^{25} = 0$

$25p^{24} - 25p^{25} - 5p^{25} = 0$

$25p^{24} - 30p^{25} = 0$

$5p^{24}(5 - 6p) = 0$

$p = 0, p = \frac{5}{6}$

As $p > 0$, $p = \frac{5}{6}$.

4 a 150 words **b** 69 words **c** 329 words

d 89 days **e** $t = e^{0.01w - 1.5} - 1$

5 **a** **i** $$0 = \frac{1}{200}(8a + 4b + c)$$
$$-0.06 = \frac{1}{200}(12a + 4b)$$
$$0 = \frac{1}{200}(48a + 8b)$$

ii Proof: see worked solutions

b **i** $M(2 + 2\sqrt{3}, 0)$, $P(2 - 2\sqrt{3}, 0)$

ii $2\sqrt{3}$ km

iii 80 m

c $k = -\frac{w}{\log_e(7)}$ **d** $w = 120$ **e** 0.2 km

6 **a** $X \sim \text{Bin}(200, 0.01)$

b 0.0517

c $[0.0006, 0.0394]$

d Given that $0.01 \in [0.0006, 0.0394]$, there is insufficient evidence to suggest that the historical data isn't relevant to current standards.

e Given that there are values of $\hat{p}$ within $[0.0006, 0.0394]$ that exceed 0.01, the department cannot conclude definitively that the targets are being met.

7 **a** $\hat{p} \sim N(0.23, 0.0210^2)$

b $(\hat{p} < 0.2) = 0.07697$

c $\frac{55}{200} = 0.275 = 27.5\%$

d $E = 0.081\,33$

e $[0.2131, 0.3369]$

f Given that $0.23 \in [0.2131, 0.3369]$, then there is insufficient evidence to suggest that the proportion of voters likely to vote for the Sustainable Energy Party in this electorate has increased.

g 1 Voters either vote for the party or not (success or failure).

2 The voters likely to vote for the Sustainable Energy Party are independent of each other. This is a reasonable assumption.

3 The probability of a voter likely to vote for the Sustainable Energy Party is the same for all voters. This is most likely not valid, as the probability may depend on other factors such as the age of the voter, occupation, socio-economic status, and/or employment status.